PHYS 2212 Lab Exercise 02: Charge Distributions & Integration

PRELIMINARY MATERIAL TO BE READ BEFORE LAB PERIOD

I. Density Functions: When we consider a charge distribution consisting of *very many* point charges, it is convenient to introduce the idea of a *charge density*, which will allow us to treat the distribution as being *continuous*. A "purist" might object to this, pointing out that ultimately, at the atomic scale, all distributions are composed of individual point charges (protons and electrons). However, if we view the distribution at a *macroscopic* scale, even a region that *we* would consider to be pointlike—e.g., a box of length 0.1 mm on a side—is of vast size in comparison to atomic scales, and would consist of tens of millions of individual point charges. In that context, treating a general charge distribution as being continuous is not all that unrealistic.

A continuous *charge density function*, then, is a means of describing how a certain amount of charge is *spread out* over some particular region. Depending upon the *type* of region that is being spread over, we have three different "categories" of density functions:

- If an amount of charge is spread throughout a three-dimension volume, we have a **volume charge density**, conventionally denoted by the symbol "**ρ**". (This is, perhaps, suggestive of the familiar *mass density* that describes distributions of matter.) A volume charge density describes the *amount of charge found charge per unit volume examined*. Thus, if you were to state that the volume density at some point were 15 μC/mm^3, you would essentially be asserting that a *tiny* box around that point (again, lets say, of dimension 0.1 mm on a side) would contain a total charge of:

$$(15 \times 10^{-6}\ \text{C/mm}^3) \times (0.1\ \text{mm})^3 = 15\ \text{nC}.$$

- If an amount of charge is spread over some two-dimensional *surface*, we have a **surface charge density**, and will use the symbol η to represent such a distribution. It is important to recognize that we are treating such a distribution as having "zero thickness"—even though, technically, there is *some* non-zero thickness to a typical "**η**". Provided, however, that the thickness "*t*" is very small compared to the area of the surface, we can neglect *t* and treat the distribution as if it were truly "2D". Surface charge density is then calculated in a manner comparable to volume charge density. In this case, however, we measure the amount of charge found *per unit area of the surface examined*.

  Keep in mind, as well, that there is no *a priori* reason why the surface in question has to be *flat*—it is completely possible to deal with a situation where charge is spread over the curved surface of a sphere or cylinder.

- If charge is spread along a *line* (i.e. a one-dimensional system), we have a **linear charge density**. Just as with surface densities, there is no absolute requirement that the "line" be *straight*. We can distribute charge along circular arcs or even weird, squiggly lines. Linear charge densities are suitable for describing wires and threads—essentially, any object whose cross-sectional area is tiny in comparison to its length. Linear charge densities are represented by the symbol "**λ**", and measure the amount of charge found *per unit of length examined*.

It is important to recognize the distinction between a distribution that is *uniform*, and one that is *non-uniform*. In a uniform distribution, the charge density takes the *same value at every point within the distribution*. This means that ρ (or η or λ) has a fixed value. Moreover, in this situation we can think of the density as describing *either* the **local** value for the "charge-to-volume ratio" (i.e, the amount of charge per unit volume found in a *tiny* region surrounding some *particular* point), or else as a **global** "charge-to-volume ratio" (i.e. an overall description of the object as a whole, in terms of the *total* amount of charge and the *total* volume of the object). However, **if a distribution is non-uniform, the density does *not* have a fixed value—it is a function that varies from one place to another,** and we write: $\rho = \rho(\boldsymbol{r})$. In such a case, the value of ρ at some particular position ***r*** describes only the local properties of the distribution, and tells us *nothing* about the distribution as a whole. Likewise, if we know the *total* charge and *total* volume of a non-uniform distribution, the ratio $Q_{\text{tot}} / V_{\text{tot}}$ would tell us ***nothing*** about the local density at any *particular* point within the distribution.

In order to put the ideas of charge density to use, we should first recognize that in the majority of situations that we will confront, **calculating electrostatic effects requires us to deal with charges on a pointlike basis**. For example, if we want to use Coulomb's Law to calculate a force or a field strength, we would have to break the distribution up into tiny pointlike charges, compute the force or field due to each charge separately, and then add up the results via superposition. Keep in mind, though that a "pointlike" charge must have *tiny* dimensions—so we must break our distribution into *infinitesimally small subregions*, in order to proceed. We would then need to know the *amount of charge* in any given subregion, and that's where the idea of a density function "comes to the rescue":

- If we have a volume distribution, then our "tiny regions" are infinitesimal volume elements dV. For some *particular* region, found at the position $\boldsymbol{r}$ (i.e. a tiny sub-volume which *contains* the point $\boldsymbol{r}$), the amount of charge found within the subregion will be the **local** charge density multiplied by the volume of the region:

$$dQ = \rho(\boldsymbol{r}) \cdot dV$$

  Here, we're using "dQ" to represent the charge because it is only an infinitesimal sub-unit of the total charge on the entire object. Note very clearly that we **must use the local value for the density**; the "global density" of the object as a whole *will not suffice*. This distinction isn't necessary, of course, if the distribution is uniform, but *students commonly overlook this requirement for* ***non-uniform*** *distributions*!

- For surface or linear distributions, the logical process is the same; we need only recognize that in these cases, the tiny subregions are either areas (dA for flat areas, dS for curved surfaces) or lengths (dx , dy, or dz for straight-line segments along the coordinate axes, ds or $d\ell$ for arc-lengths along a curved distribution). So, in these cases, we would write something like;

$$dQ = \eta(\boldsymbol{r}) \cdot dA$$

$$dQ = \lambda(\boldsymbol{r}) \cdot d\ell$$

One can then proceed with the steps of calculating an expression for the desired electrostatic quantity (e.g. force or field), and then using the principle of superposition to sum over all the pointlike charges. However, since we have broken our distribution into sub-regions of *infinitesimal size*, we will have to have an *infinite number of subregions* in order to "cover" the whole object! In that case, the conventional notion of "summation" has to be replaced by the process of *integration*. **In the context of this course, "integration" is a computational mechanism for determining the value of a sum in the limit where an infinitely large number of infinitesimally small things are being added.**

II. Charge Density ⇔ Total Charge: Let us now put the conceptual ideas of the preceding section to use. Suppose that we have some volume charge density, and we wish to know how much *total* charge it contains. *If and only if the distribution is uniform*, we could determine Q_{tot} by simple multiplication:

$$Q_{\text{tot}} = \rho \cdot V_{\text{tot}} \quad (\textbf{only} \text{ if } \rho = \textit{constant} \text{ throughout the object!})$$

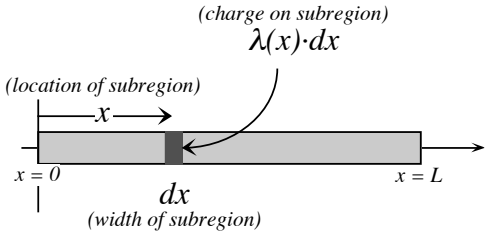
More often than not, we will be faced with a non-uniform distribution, and will have to try something else. We break the object up into subregions ΔV_i, compute the charge in each subregion $\Delta Q_i = \rho_i \cdot \Delta V_i$, and sum. Of course, for this to work, we must make sure that the subregions are locally small enough that the value of ρ does not vary noticeably over region "i"—and that means that the regions should really be infinitesimals dV :

$$Q_{tot} = \lim_{N \to \infty} \sum_{i=1}^{N} \rho_i \cdot \Delta V_i \to \int \rho(\mathbf{r})\, dV$$

In order for this integration to succeed, it is imperative that you recognize that **one cannot simply integrate the density over the *coordinates*; for the process to work, one must first be able to write an expression for the *differential volume dV* in terms of the coordinates.** Thus, a key step in the analytical process is to think carefully about the *type* of volume region which is most suitable for a given charge distribution, and to analyze

the geometry of that region in order to get a proper expression for the differential volume (or differential area, or differential length, as the case may be.

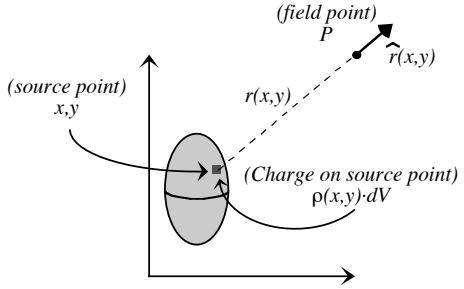
As a brief example: suppose we were given a thin rod along the x–axis, with a non-uniform linear charge density $\lambda(x)$, extending from $x = 0$ to $x = L$. In this case, the "suitable subregion" would be a tiny sub-length along the x–axis. Since it's "tiny", we'll call it an infinitesimal dx, and if we wanted to know the *total* charge on the thread, we would add the charge on various subregions:



$$Q_{tot} = \int_0^L \lambda(x)\, dx$$

Note that the correct result *cannot* be obtained any other way—for example, by multiplying $\lambda(L) \cdot L$, or by trying to be clever and multiplying $\lambda(L/2) \cdot L$.

III. Field Calculations: A similar process can be applied in order to determine the electric field due to a distribution of charge. However, we must recognize that the summation in this case is not simply over the *charge*, but over the *field vectors that are created by each of the pointlike charges in the distribution*. That is, for a given distribution, we pick a particular field point (i.e. an empty point in space at which we are trying to find $\boldsymbol{E}_{\text{net}}$). We then break the distribution into pointlike subregions, with infinitesimal charges $dQ = \rho(\boldsymbol{r}) \cdot dV$. Keep in mind that $\boldsymbol{r}$ represents a generic point *which is somewhere inside the distribution of charge*—(the so-called "source point"). The electric field *due to just this **one** source point* is:



$$d\vec{E} = k\frac{dQ}{r^2}\hat{r}$$

We've used "$d\vec{E}$" to represent the field because it is only an infinitesimal sub-portion of the *net* field. In the denominator, "r" represents the distance from the source point to the field point, and is (in general) a quantity that depends upon the location of the source point (as well as the chosen field point). Thus, r is something that must be expressed *as a function of the source point coordinates*. Similarly, dQ should be expressed as: (density function evaluated at source point coordinates) times (infinitesimal size of region containing source point), and the unit vector $\hat{r}$—which represents "unit vector *at the field point* which points directly *away from* the source point"—should be expressed as a function of the source point coordinates. All these steps require careful geometry—including some basic vector analysis (for $\hat{r}$). The end result is (hopefully) an algebraic expression for the (vector) electric field contribution $d\vec{E}$ due to a generic source point, **expressed entirely as a function of source point coordinates**. The process of summing all field contributions is then one of integrating a function over the volume of the distribution—albeit, a *vector-valued* function:

$$\vec{E}_{net} = \int d\vec{E} = \int k\frac{dQ}{r^2}\hat{r}$$

Generally speaking, *the only thing that is 100% guaranteed to factor out of the integral is the electrostatic constant "k"!!* As a final cautionary note regarding this integration process: anyone who attempts to work such a problem without a carefully drawn sketch has ***little or no chance*** of getting the geometry right!