PHYS 2212 Problem-Solving Studio 08

Mar 28–31

SOLUTION

Hyperfine Splitting

In your Astrophysics Laboratory course, you are performing experiments in microwave spectroscopy. Your instructor mentions something called “hyperfine structure”, that can split spectral lines in the sub-millimenter wavelengths of radio waves that arrive from interstellar space. Looking up the term on the internet, you find that hyperfine splitting occurs in an atom when the magnetic field of an orbiting electron interacts with the magnetic field of the atomic nucleus. Out of curiousity, you decide to calculate the strength of the magnetic field created by an electron at the location of the central proton, in a hydrogen atom. You model the electron as orbiting in a perfect circle around the proton, at the Bohr radius, $r_{\text{B}} = 5.29 \times 10^{-11}$ m.

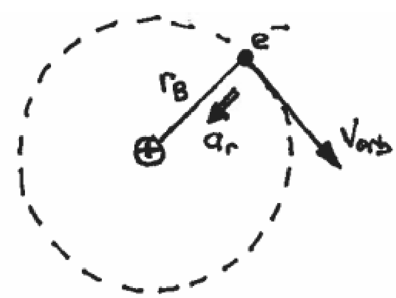
Instructions:

Construct a visual representation of the situation described, with all physical quantities represented by symbolic variables. Identify the concepts that will be needed to answer the question posed, as well as any simplifying assumptions that you will use. Outline a plan (that is, a series of analytical steps) that you will use solve the problem, and then follow those steps to solve the problem.

You may work as a group to complete this exercise, but each student is expected to submit an individual solution.

Assumptions

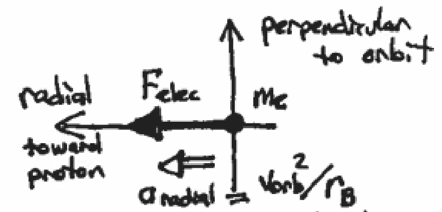
- Electron is in uniform circular motion around proton
- radial force is provided by coulomb attraction
- circular-motion analysis will give us orbital speed
- relativistic or quantum "weirdness" can be ignored



Top view of orbit

Problem Formulation

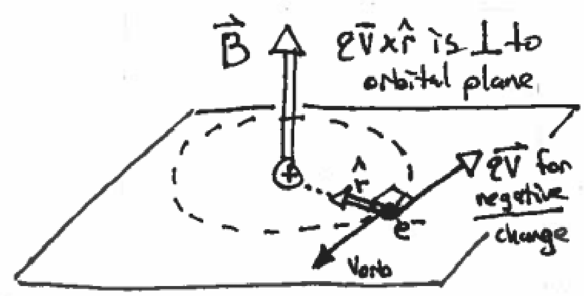
- Determine the orbital speed of an electron orbiting a proton at the Bohr radius, and use that speed in the Biot-Savart Law to find the magnetic field created at the center of the electron's orbit



side-view free body diagram

Outline of solution

1. Circular 2nd Law : $\Sigma \vec{F}_{radial} = m\vec{a}_{radial}$
2. Biot-Savart Law : $\vec{B} = \frac{\mu_0}{4\pi}\frac{(q\vec{v})\times\hat{r}}{r^2}$
   - Right Hand Rule needed for cross product
   - only magnitude of field is required

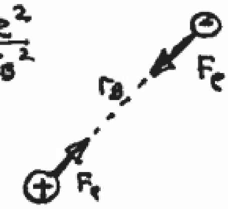


perspective view of orbital plane

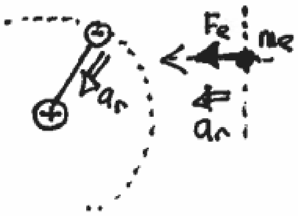
$\vec{B}$ is directed along $(q\vec{v})\times\hat{r}$

① Coulomb force between proton and electron is attractive, $F_e = \frac{ke^2}{r_B^2}$

(Proton is much more massive, and basically doesn't move)



② This force provides radial acceleration in 2^{nd} law



$\Sigma \vec{F}_r = m\vec{a}_r$

$+\frac{ke^2}{r_B^2} = m_e\left(\frac{V_{orb}^2}{r_B}\right) \rightarrow \boxed{V_{orb} = \sqrt{\frac{ke^2}{m_e r_B}} = 2.19 \times 10^6\ m/s}$

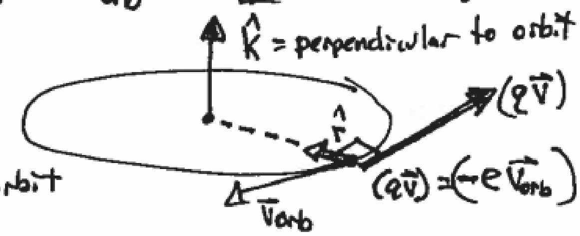
$(m_e = 9.11 \times 10^{-31}\ kg)$

③ Biot-Savart law: $\vec{B} = \frac{\mu_0}{4\pi}\frac{(q\vec{v}) \times \hat{r}}{r^2}$ $\hat{r}$ = unit vector from electron to proton

- Note that electron charge is negative, so $q\vec{v} = -e\vec{V}_{orb}$ is opposite to $\vec{V}_{orb}$!
- Note that $q\vec{V}_{orb}$ = tangential, $\hat{r}$ = radial] vectors are $\perp$ to each other

$\Rightarrow$ cross product $q\vec{V}_{orb} \times \hat{r} = eV_{orb}\hat{k}$

where $\hat{k}$ = direction $\perp$ to plane of orbit



So, putting it all together:

$\vec{B} = \frac{\mu_0}{4\pi}\frac{eV_{orb}}{r_B^2}\hat{k} \rightarrow |\vec{B}| = \frac{\mu_0}{4\pi}\frac{eV_{orb}}{r_B^2}$ $\boxed{B = 12.5\,T}$

this is about 200,000 × Earth's magnetic field!