PHYS 2212 Problem-Solving Studio 03

Feb 07–10

SOLUTION

Gausss Law Adaptation

You are watching an episode of Doctor Who, in which the thirteenth Doctor is falling down a mineshaft drilled from the surface of a planet all the way to its center. As she falls, she scans with her sonic scredriver, and reassures her Companions that, since the density of the planet decreases linearly to zero at the center, they are in no danger because their acceleration will also decrease linearly toward zero at the center. Watching the show, you are concerned that perhaps the writers of the programme did not get the physics right. Recalling that the Law of Gravitation is mathematically identical to Coulombs Law—and having just studied Gausss Law in your Electromagnetism class—you realize that a gravitational analog of Gauss' Law will help you decide.

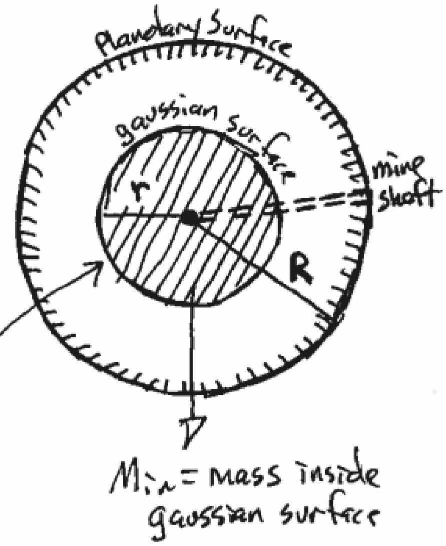
	Gravitation	**Electrostatics**
Force:	$\vec{F}_g = -G\frac{Mm}{r^2}\hat{r}$	$F_e = \frac{1}{4\pi\epsilon_0}\frac{Qq}{r^2}\hat{r}$
Field:	$\frac{\vec{F}}{m} = \vec{a}_g = -G\frac{M}{r^2}\hat{r}$	$\frac{\vec{F}}{q} = \vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}$
Gauss' Law:	$\oint \vec{a}_g \cdot d\vec{A} = ???$	$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{in}}}{\epsilon_0}$

Instructions:

Construct a visual representation of the situation described, with all physical quantities represented by symbolic variables. Identify the concepts that will be needed to answer the question posed, as well as any simplifying assumptions that you will use. Outline a plan (that is, a series of analytical steps) that you will use solve the problem, and then follow those steps to solve the problem.

You may work as a group to complete this exercise, but each student is expected to submit an individual solution.

- Assume spherically symmetric planet, with "linear density" ρ = max at surface, 0 at center
  $\Rightarrow \rho(s) = A \cdot s$ where $0 \le s \le R$, A = constant
- Gravitational analog of $\frac{\vec{F}_e}{q} = \vec{E}$ is $\frac{\vec{F}_g}{m} = \vec{a}_g$
  $\Rightarrow$ "grav. field" is a field of acceleration vectors so "grav. flux" is $\Phi_g = \oint \vec{a}_g \cdot \delta\vec{A}$
- invoking spherical symmetry $\rightarrow$ $\boxed{\Phi_g = a_g(r) \cdot 4\pi r^2 \text{ for a spherical gaussian surface}}$



Problem formulation: Apply Grav. Gauss's Law to determine acceleration as a function of position, inside a planet having a spherical linear density function, $\rho(s) = AS$

<u>Plan</u>:
1. Complete the electricity $\rightarrow$ gravity analog of Gauss's law
2. Determine appropriate M_{in} by <u>integrating</u> $\rho(s)$ over shells, $0 \le s \le r$
3. Substitute and solve for acceleration function $a_g(r)$
4. Compare to stated assumption, evaluate accuracy of writers' claim

① First, we need to write Gauss's law for gravity: $\oint \vec{E}\cdot d\vec{A} = Q_{in}/\varepsilon_0 \longrightarrow$???

a] analog: $\vec{E}$ becomes $\vec{a}_g$

b] analog: Q_{in} becomes M_{in}

① c] analog: $1/\varepsilon_0$ becomes ???

compare Force laws: $\frac{Q}{4\pi\varepsilon_0 r^2} \leftrightarrow G\frac{M}{r^2}$

so $\frac{1}{4\pi\varepsilon_0} \leftrightarrow G$ or $\boxed{\frac{1}{\varepsilon_0} \leftrightarrow 4\pi G}$

So, our Gauss's Law is $\boxed{\oint \vec{a}_g \cdot \delta\vec{A} = -4\pi G M_{in}}$ negative as positive mass causes inward field

$\longrightarrow a_g(r)\cdot \cancel{4\pi} r^2 = -\cancel{4\pi} G M_{in}$ where M_{in} = all mass found at coordinates s that are $\leq r$

(for spherical GS)

② Summing mass means integrating over shells:

$dM = \rho\,\delta V = \rho(s)\cdot \underbrace{4\pi s^2 ds}$ = mass of one thin shell

$\hookrightarrow$ shell volume = surface area x thickness

shell somewhere inside gaussian sphere: $s \leq r$

so $M_{in} = \int_{s=0}^{s=r} (As)4\pi s^2 ds = 4\pi A \int_0^r s^3 ds = 4\pi A\left[\frac{r^4}{4}\right]$

integrate from center to gaussian surface and STOP

$\Longrightarrow \boxed{M_{in} = \pi A r^4}$

③ Subbing everything together:

$a_g(r)\cdot \cancel{4\pi} r^2 = -\cancel{4\pi} G\left[\pi A r^4\right]$

$\boxed{a_g(r) = -\pi G A r^2}$

Writers were partially correct
- a_g does $\rightarrow 0$ at center

but
- a_g is quadratic, not linear

$a_g(\text{center}) = 0$

$a_g(\text{surface}) = \pi G A R^2$

magnitude