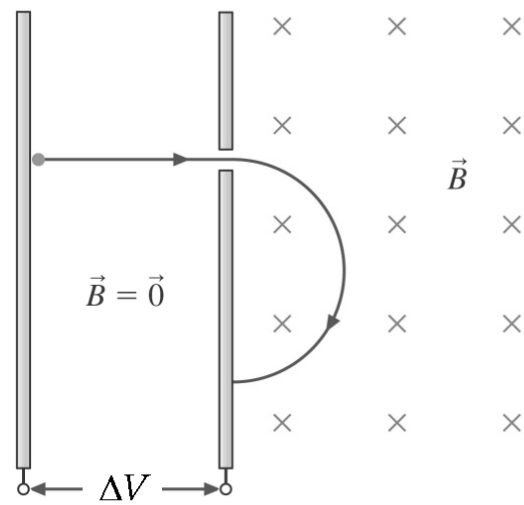
I. (16 points)A particle is accelerated from rest through a potential difference, then passes through a region of uniform magnetic field with magnitude B, which is directed into the page. It follows a half-circle, as illustrated, and strikes the plate defining the edge of the field.

If the particle has mass m and charge magnitude q, at what time Δt after entering the field does the particle strike the plate? Express your answer in terms of parameters defined in the problem, and physical or mathematical constants.



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Use Newton's Second Law. While the particle is in the magnetic field, the only force on it is the magnetic force. The path of the particle is a segment of a circle, so the magnetic force causes a centripetal acceleration.

$$\sum F_c = F_B = ma_c \qquad \Rightarrow \qquad qvB\sin\phi = m\frac{v^2}{r}$$

The velocity is always perpendicular to the field, so $\phi = 90°$ and $\sin\phi = 1$. The magnitude of the velocity is the path through which the particle has moved, divided by the time required, $v = s/\Delta t$.

$$qvB\sin\phi = m\frac{v^2}{r} \qquad \Rightarrow \qquad qB = m\frac{s/\Delta t}{r}$$

The particle strikes the plate after moving through a half circle, so $s = \pi r$.

$$qB = m\frac{\pi r}{r\,\Delta t} \qquad \Rightarrow \qquad \Delta t = \frac{\pi m}{qB}$$

II. (16 points) In the illustrated electric circuit, $\mathcal{E}_1 = 9.0\,\mathrm{V}$, $\mathcal{E}_2 = 12\,\mathrm{V}$, $R_1 = 11\,\Omega$, $R_2 = 22\,\Omega$, and $R_3 = 33\,\Omega$. What power is dissipated in the resistor R_2?

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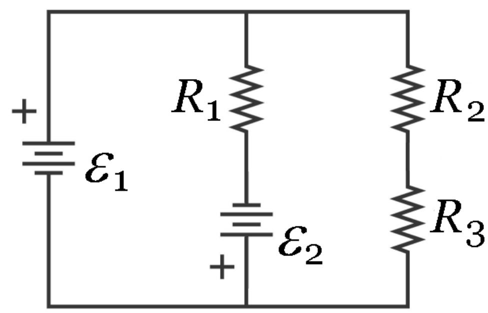
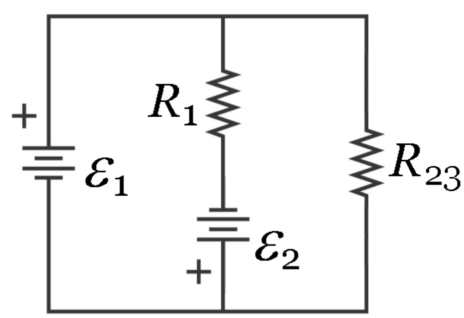


Note that R_2 and R_3 are in series, so their equivalent resistance, R_{23}, is $R_{23} = R_2 + R_3$. Redrawing the circuit, one sees that the potential difference ΔV_1 across R_{23} is $\mathcal{E}_1$, so the current through R_{23} can be found:

$$\Delta V_1 = I_{23} R_{23} \qquad \Rightarrow \qquad I_{23} = \mathcal{E}_1 / R_{23} = \frac{\mathcal{E}_1}{R_2 + R_3}$$

Since R_2 and R_3 are in series, this is also the current through R_2. The electric power dissipated in any device is $P = I\,\Delta V$. For a resistor, $\Delta V = IR$. So

$$\begin{aligned} P_2 &= I_2\,\Delta V_2 = I_2\,(I_2 R_2) = I_2^2 R_2 \\ &= \left(\frac{\mathcal{E}_1}{R_2 + R_3}\right)^2 R_2 = \left(\frac{9.0\,\mathrm{V}}{22\,\Omega + 33\,\Omega}\right)^2 (22\,\Omega) = 0.59\,\mathrm{W} \end{aligned}$$



1. (6 points) In the problem above, let the power dissipated in resistor R_2 be P_2. If the emf of both batteries were *doubled*, what new power P_2' would be dissipated in resistor R_2?

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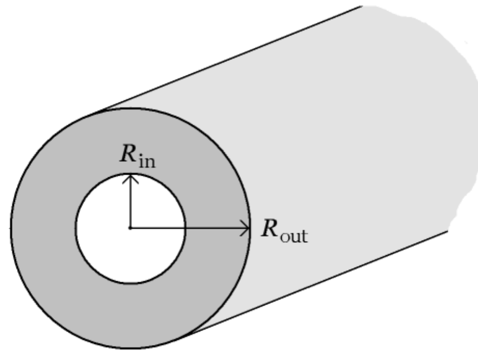
The potential across the resistor must be due to the emfs of the batteries. Without solving the problem above, we don't know exactly *how* the potential depends on the emfs. But since emf and potential difference both have the same dimensions (SI units of Volts), doubling all the emfs *must* double the potential difference across the resistor. As the power dissipated in a resistor depends on the square of the potential difference across it, $P = (\Delta V)^2 / R$,

$$P_2' = 4P_2$$

III. (16 points)A hollow cylindrical conductor has inner radius R_{in} and outer radius R_{out}. It carries a current out of the page whose density, $\vec{J}$, has magnitude

$$|\vec{J}| = J_0 \left(\frac{R_{\text{in}} R_{\text{out}}}{r^2} \right)$$



where r is the distance from the cylinder axis and J_0 is a constant. What is the magnitude of the magnetic field at a distance R' from the cylinder axis, where $R_{\text{in}} < R' < R_{\text{out}}$? Express your answer in terms of parameters defined in the problem, and physical or mathematical constants.

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Use Ampere's Law.

$$\oint \vec{B} \cdot \vec{ds} = \mu_0 I_{\text{thru}}$$

Choose a circular Amperian Loop of radius R'. The field is parallel to $\vec{ds}$ at every point around the loop.

$$\oint \vec{B} \cdot \vec{ds} = \oint B \cos 0^\circ \, ds = B \, 2\pi R'$$

The current though this Amperian Loop must be found from the current density, $I_{\text{thru}} = \int \vec{J} \cdot \vec{dA}$. Choose an area element that is a thin ring of radius r and width dr. Add up all the rings from R_{in} to R' (integrate). The current density is parallel to the area vector in each ring.

$$I_{\text{thru}} = \int \vec{J} \cdot \vec{dA} = \int_{R_{\text{in}}}^{R'} J_0 \left(\frac{R_{\text{in}} R_{\text{out}}}{r^2} \right) \cos 0^\circ \, 2\pi r \, dr = 2\pi J_0 R_{\text{in}} R_{\text{out}} \int_{R_{\text{in}}}^{R'} \frac{dr}{r}$$

$$= 2\pi J_0 R_{\text{in}} R_{\text{out}} \ln r \Big|_{R_{\text{in}}}^{R'} = 2\pi J_0 R_{\text{in}} R_{\text{out}} \left(\ln R' - \ln R_{\text{in}} \right) = 2\pi J_0 R_{\text{in}} R_{\text{out}} \ln \frac{R'}{R_{\text{in}}}$$

Putting these results together

$$B \, 2\pi R' = \mu_0 2\pi J_0 R_{\text{in}} R_{\text{out}} \ln \frac{R'}{R_{\text{in}}} \qquad \Rightarrow \qquad B = \frac{\mu_0 J_0 R_{\text{in}} R_{\text{out}}}{R'} \ln \frac{R'}{R_{\text{in}}}$$

2. (6 points) Let your answer to the problem above be B'. What is the magnitude of the magnetic field at a distance $R'' = R_{\text{in}}/2$ from the cylinder axis?

. . . .

No current passes through an Amperian Loop of radius $R_{\text{in}}/2$.

Zero

3. (7 points) The current loop in the figure is shown in cross-section, with current flowing into the page at the top, and out at the bottom. A permanent magnet is to the left of the loop, with the north end nearest the loop. If there is a force on the loop from the magnet, what is its direction?

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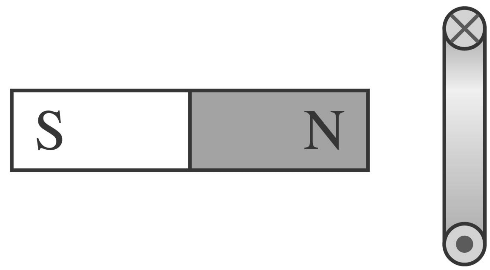

The current in the loop produces a magnetic field. One may find the direction of this field in the center of the loop with the Biot-Savart Law for current

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{\vec{d\ell} \times \hat{r}}{|\vec{r}|^2}$$

or the Right-Hand-Rule shortcut, in which the fingers of the right hand are curled around the loop in the direction of current flow. The thumb shows the direction of the magnetic field produced in the center of the loop, which is to the left in this situation. Since magnetic field lines emerge from a North pole, the magnetic dipole created by the loop has a North pole to the left. This is repelled by the North pole of the permanent magnet, so

There is a force on the loop to the right.

4. (7 points) A charge $+q_0 > 0$ moves with velocity $\vec{v} = v\hat{x}$ along a line $y = +d, z = 0$. A second charge $-q_0$ travels with identical velocity $\vec{v}$ along a line $y = -d, z = 0$. The two charges pass the origin at the same instant. When the charges pass the origin, which of the following expressions best describes the magnetic field at the origin?

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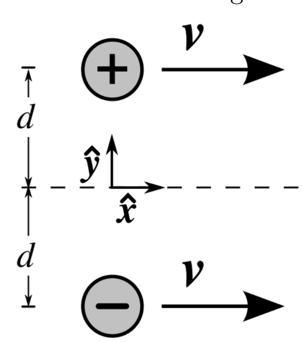


Use the Biot-Savart Law,

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{q\vec{v} \times \hat{r}}{|\vec{r}|^2}$$

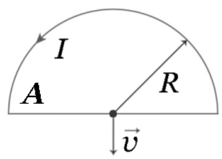
Each charge generates field $\frac{\mu_0}{4\pi} \frac{q_0 v}{d^2}$ into the page, so the total is

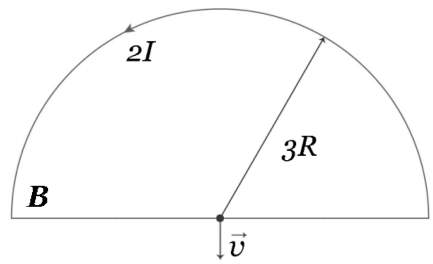
$$-\frac{\mu_0}{2\pi} \frac{q_0 v}{d^2} \hat{z}$$

5. (7 points) The wires shown below are bent into semicircles, connected by a straight wire between their ends. Wire A is on the left and wire B is on the right, and the two wires are far apart. An electron is located at the center of each circle (but outside the wires), moving with velocity $\vec{v}$ in the direction perpendicular to the straight connecting lines, as shown. Compare the magnitude of the force F_B on the electron at the center of wire B, to the magnitude of the force F_A on the electron at the center of wire A.

. . . .

The magnetic field at the center of each loop, where the electron is located, can be found using the Biot-Savart Law.





$$d\vec{B} = \frac{\mu_0 i}{4\pi}\frac{\vec{ds}\times\hat{r}}{r^2}$$

$$dB = \frac{\mu_0 i}{4\pi}\frac{ds\sin\phi}{r^2} \qquad \text{so} \qquad B = \int dB = \int_0^{\pi r}\frac{\mu_0 i\, ds}{4\pi r^2} = \frac{\mu_0 i}{4\pi r^2}\int_0^{\pi r} ds = \frac{\mu_0 i}{4\pi r^2}(\pi r - 0) = \frac{\mu_0 i}{4r}$$

as ϕ is 90° for the magnetic field due to the curved parts of the wires, and zero or 180° for the straight segment. This means the straight segment does not contribute to the magnetic field at the location of the electron. From the cross-product (right-hand rule), the magnetic field is out of the page.

The magnetic force magnitude on the electron can be determined

$$\vec{F} = q\vec{v}\times\vec{B} \qquad \Rightarrow \qquad F_A = ev_0\frac{\mu_0 I}{4R} \qquad \text{and} \qquad F_B = ev_0\frac{\mu_0(2I)}{4(3R)} = \frac{2}{3}ev_0\frac{\mu_0 I}{4R}$$

So

$$F_B = \frac{2}{3}F_A$$

6. (7 points) A long straight wire carries a current out of the page, as illustrated on the near-right. The small current loop, shown in cross-section on the far-right, carries a current into the page at the top and out of the page at the bottom. The loop is free to rotate, but not to translate. What is the stable equilibrium orientation, if any, of the loop?

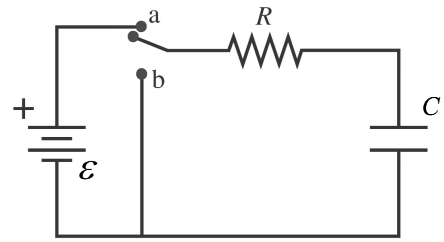
. . . .

In the original orientation, the small loop makes magnetic field to the left through its center. Thus, its magnetic moment is originally to the left. The loop will rotate to align its magnetic moment with the field created by the wire. That field is up the page at the location of the small loop.

7. (6 points) The switch in the illustrated circuit is set to position "b" for a long time, then set to position "a" for a time t_a, then set back to position "b". After that, the charge on the capacitor is

$$Q = Q_0 e^{-t_b/RC}$$

where t_b is the time from returning the switch to position "b". What is Q_0?



. . . .

Charge on the capacitor remains constant as the switch is thrown from "a" to "b". At the end of time t_a, that charge is

$$Q_0 = Q(t_a) = Q_\infty \left(1 - e^{-t_a/RC}\right) = C\mathcal{E}\left(1 - e^{-t_a/RC}\right)$$

8. (6 points) In the problem above, after the switch is set back to position "b", the current through the resistor is

$$I = I_0 e^{-t_b/RC}$$

where t_b is the time from returning the switch to position "b". What is I_0?

. . . .

Charge on the capacitor remains constant as the switch is thrown from "a" to "b". At the end of time t_a, that charge is

$$Q(t_a) = Q_\infty \left(1 - e^{-t_a/RC}\right) = C\mathcal{E}\left(1 - e^{-t_a/RC}\right)$$

So

$$\Delta V_C(t_a) = Q(t_a)/C = \mathcal{E}\left(1 - e^{-t_a/RC}\right)$$

Once the switch is thrown back to "b", the potential difference across the resistor must be the same as the potential difference across the capacitor.

$$I = \Delta V_R/R \quad \Rightarrow \qquad I_0 = \Delta V_C(t_a)/R = (\mathcal{E}/R)\left(1 - e^{-t_a/RC}\right)$$