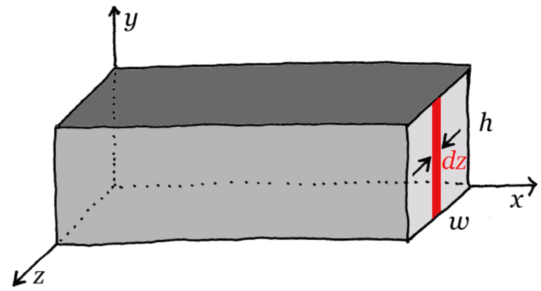

I. (16 points) An infinite straight wire with rectangular cross-section lies with one edge on the x axis, a segment of which is shown in the figure. It has height h in the y direction, and carries a non-uniform current density $\vec{J}$ that varies according to

$$\vec{J} = J_0 \left(\frac{z}{h}\right) \hat{\imath}$$

where J_0 is a positive constant. If the wire carries a total current I_0, what is the width w of the wire in terms of parameters defined the problem and physical or mathematical constants?



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The current can be related to the current density by

$$I = \int \vec{J} \cdot d\vec{A} = \int J \cos\theta \, dA$$

The current density depends on z, so choose an area element dA that is "small" in the z direction. The element shown has height h, width dz, and has an area vector in the $+\hat{\imath}$ direction, so θ is zero and $\cos\theta = 1$. Add up (integrate) the current through all the strips from $z = 0$ to $z = w$.

$$I_0 = \int_0^w J_0 \left(\frac{z}{h}\right) h \, dz = J_0 \left[\frac{z^2}{2}\right]_0^w = J_0 \frac{w^2}{2} \qquad \Rightarrow \qquad w = \sqrt{2I_0/J_0}$$

1. (6 points) A parallel-plate capacitor has plates of area $A = 0.25\,\mathrm{m}^2$ and separation $s = 2.4\,\mathrm{mm}$. A battery charges it to a potential difference of $845\,\mathrm{V}$, and is then disconnected. A $2.55\,\mathrm{kg}$ slab of titanium dioxide (dielectric constant 110) is shaped to exactly fill the gap between the plates. The slab will be released from rest outside the capacitor. What remains the same as the slab moves into the gap between the plates?

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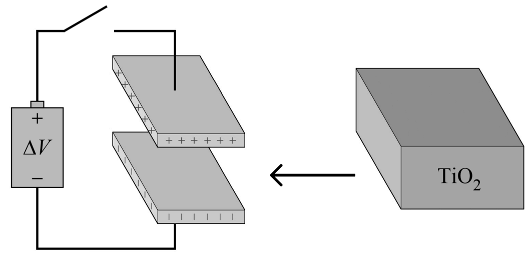

Since the capacitor has been disconnected from the battery, the charge cannot change. When the dielectric is between the plates, it has an induced field, reducing the field between the plates. Since the distance between the plates does not change, and reduced field means that the potential difference between the plates is reduced, as well. The only thing (among the choices) that remains the same is

The charge magnitude on each capacitor plate.

II. (16 points) In the problem above, what maximum speed does the slab have as it passes between the plates?

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Use the Work-Energy Theorem.

$$W_{\text{ext}} = \Delta K + \Delta U + \Delta E_{\text{th}}$$

Choose a system consisting of the capacitor and the slab. With that choice, there are no external forces doing work. There are also no internal dissipative forces changing that system's thermal energy. The kinetic energy of the system changes, as the slab's speed changes. The potential energy of the system changes, as the electric field in the capacitor changes (or, from another perspective, the capacitance of the capacitor changes).

$$0 = \left(\tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2\right) + \left(\frac{Q^2}{2C_f} - \frac{Q^2}{2C_i}\right) + 0$$

where the expression $U_C = Q^2/2C$ was chosen as the charge on the capacitor plates remains constant. The initial speed of the slab is zero. The final capacitance is $C_f = \kappa C_i$.

$$0 = \tfrac{1}{2}mv_f^2 + \left(\frac{Q^2}{2\kappa C_i} - \frac{Q^2}{2C_i}\right) \qquad \Rightarrow \qquad \tfrac{1}{2}mv_f^2 = \frac{Q^2}{2C_i}\left(1 - \frac{1}{\kappa}\right)$$

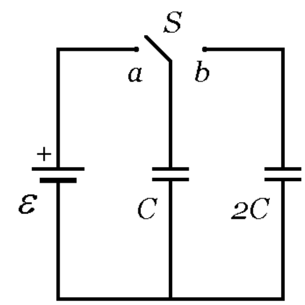
The charge on the capacitor plates can be found from the initial potential difference and the definition of capacitance, $Q = C_i\,\Delta V$.

$$\tfrac{1}{2}mv_f^2 = \frac{(C_i\,\Delta V)^2}{2C_i}\left(1 - \frac{1}{\kappa}\right) = \tfrac{1}{2}C_i\,(\Delta V)^2\left(1 - \frac{1}{\kappa}\right) = \tfrac{1}{2}\epsilon_0\frac{A}{d}\,(\Delta V)^2\left(1 - \frac{1}{\kappa}\right)$$

where the capacitance of a parallel-plate capacitor, $C = \epsilon_0 A/d$, has been substituted. Note that $d = s$. Solve for the final speed.

$$v_f = \Delta V\sqrt{\frac{\epsilon_0 A}{sm}\left(1 - \frac{1}{\kappa}\right)} = (845\,\mathrm{V})\sqrt{\frac{\left(8.854\times10^{-12}\,\frac{\mathrm{C}^2}{\mathrm{N\cdot m^2}}\right)(0.25\,\mathrm{m}^2)}{(2.4\times10^{-3}\,\mathrm{m})\,(2.55\,\mathrm{kg})}\left(1 - \frac{1}{110}\right)} = 0.016\,\mathrm{m/s}$$

III. (16 points) The emf of the battery in the illustrated circuit is $\mathcal{E}$. The capacitor on the left has capacitance C and that on the right has capacitance $2C$. The switch S is placed in position a for a long time, then thrown to position b. Once equilibrium is reached, and with respect to zero energy stored in an uncharged capacitor, what is the energy stored in the capacitor on the right? Express your answer in terms of parameters defined in the problem and physical or mathematical constants.



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When the switch is in position a, only the capacitor C is connected to the battery. The charge on this capacitor can be found from the definition of capacitance.

$$Q = C\,\Delta V \qquad \Rightarrow \qquad Q_a = C\mathcal{E}$$

When the switch is thrown to position b, this total charge is redistributed between the two capacitors until the electric potential difference across each capacitor is the same.

$$Q_b = Q_a = C\mathcal{E} = Q_C + Q_{2C} \qquad \text{and} \qquad \Delta V_C = \Delta V_{2C} \qquad \Rightarrow \qquad \frac{Q_C}{C} = \frac{Q_{2C}}{2C}$$

Since we're interested in the capacitor $2C$ on the right, eliminate the charge on the capacitor C on the left.

$$Q_C = Q_b - Q_{2C} \qquad \Rightarrow \qquad \frac{Q_b - Q_{2C}}{C} = \frac{Q_{2C}}{2C}$$

Solve for Q_{2C}.

$$2Q_b - 2Q_{2C} = Q_{2C} \qquad \Rightarrow \qquad 2Q_b = 3Q_{2C} \qquad \Rightarrow \qquad Q_{2C} = \frac{2}{3}Q_b = \frac{2}{3}C\mathcal{E}$$

In general, the energy stored in a capacitor is related to its charge by $U = \frac{Q^2}{2C}$. In this case, the capacitance is $2C$, so

$$U = \frac{Q_{2C}^2}{2\,(2C)} = \frac{\left(\frac{2}{3}C\mathcal{E}\right)^2}{4C} = \frac{C\mathcal{E}^2}{9}$$

2. (6 points) In the problem above, how does the total energy stored in both capacitors while the switch is in position a compare to the total energy stored in both capacitors when the switch is in position b?

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Since the charge is redistributed when the switch is thrown to b, then energy in the system must be less when the switch is thrown to b. Charge would not spontaneously move to a higher-energy or equal-energy configuration.

The total energy with the switch in position a is **greater than** the total with the switch in position b.

3. (7 points) An ideal parallel-plate capacitor is attached to a battery and allowed to achieve electrostatic equilibrium. Insulating handles are then used to push the plates closer together *while the capacitor remains connected to the battery*, until their spacing is half the original spacing. How does the potential energy stored in the capacitor with this new spacing, U_{new}, compare to that stored in the capacitor with the original spacing, U_0? (Let the potential energy stored in the uncharged capacitor be zero.)

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The capacitance of a parallel-plate capacitor is $C = \epsilon_0 \frac{A}{d}$. Since $d_{\text{new}} = d_0/2$, $C_{\text{new}} = 2C_0$. The electric potential across the capacitor must remain constant, as it remains connected to the battery.

$$U_0 = \tfrac{1}{2} C_0 (\Delta V)^2 \qquad \text{and} \qquad U_{\text{new}} = \tfrac{1}{2} C_{\text{new}} (\Delta V)^2 = \tfrac{1}{2} 2C_0 (\Delta V)^2 = 2\left[\tfrac{1}{2} C_0 (\Delta V)^2\right] = 2U_0$$

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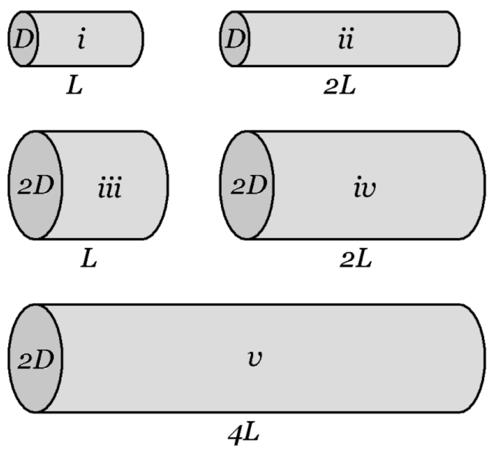
4. (7 points) When a particular potential difference ΔV_0 is applied across the length of each of the wire segments shown, the same current I_0 flows through each segment. Segment i has length L and diameter D. Rank the resistivities of the segments from *greatest* to *least*.



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From the definition of resistance ("Ohm's Law")

$$\Delta V = IR$$

one can conclude that all the segments have the same resistance R. Resistance can be related to resistivity ρ by

$$R = \rho \frac{L}{A} \qquad \Rightarrow \qquad \rho = R\frac{A}{L}$$

Calculate the resistivity of each segment.

$$\begin{aligned}
\rho_i &= R\frac{A_1}{L_1} \\
\rho_{ii} &= R\frac{A_2}{L_2} = R\frac{A_1}{2L_1} = \rho_i/2 \\
\rho_{iii} &= R\frac{A_3}{L_3} = R\frac{4A_1}{L_1} = 4\rho_i \\
\rho_{iv} &= R\frac{A_4}{L_4} = R\frac{4A_1}{2L_1} = 2\rho_i \\
\rho_v &= R\frac{A_5}{L_5} = R\frac{4A_1}{4L_1} = \rho_i
\end{aligned}$$

So

$$iii > iv > i = v > ii$$

5. (7 points) The illustrated wire is composed of two segments, 1 and 2, having different conductivities σ_1 and σ_2, and different radii R_1 and R_2. The conductivity of segment 2 is only half the conductivity of segment 1. The radius of segment 2 is three times the radius of segment 1. What is the ratio of the electric field magnitude in segment 1 to that in segment 2, E_1/E_2?

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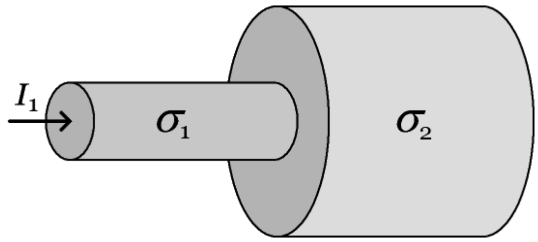

The electric field can be related to the current density and the current. In terms of magnitudes

$$J = \sigma E = \frac{I}{A}$$

The current in each segment is the same.

$$I_1 = I_2 \qquad \Rightarrow \qquad \sigma_1 E_1 A_1 = \sigma_2 E_2 A_2$$

So

$$\frac{E_1}{E_2} = \frac{\sigma_2 A_2}{\sigma_1 A_1} = \frac{\sigma_2 \pi r_2^2}{\sigma_1 \pi r_1^2} = \frac{\sigma_2}{2\sigma_2}\left(\frac{3r_1}{r_1}\right)^2 = \frac{9}{2}$$

6. (7 points) The two circuits shown are constructed of two identical batteries and three identical light bulbs. All the bulbs are glowing. Compare the potential differences ΔV across the terminals of the two batteries, and the currents I supplied by the two batteries.

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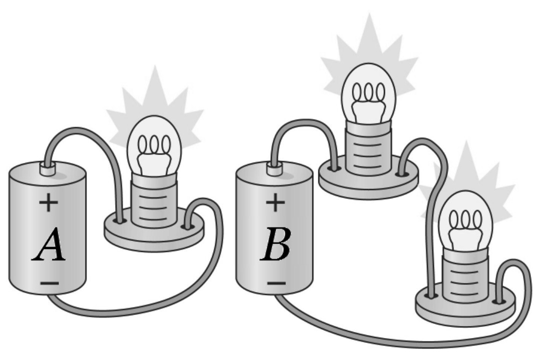


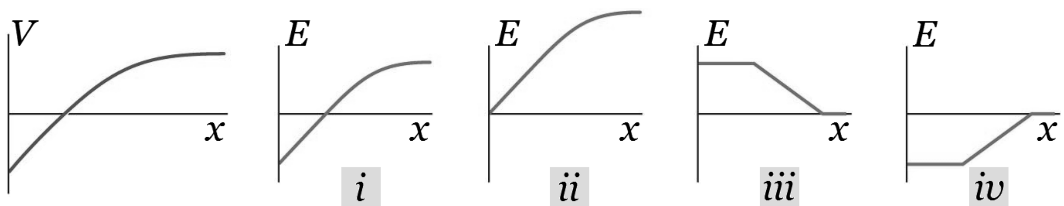
If the batteries are ideal, then since they are identical, the potential differences across their terminals will be the same. The current from battery B goes through both bulbs one after the other ("in series") so the total resistance of that circuit is greater than that for battery A. Therefore, from the definition of resistance $\Delta V = IR$, since the potential differences are the same but circuit B has greater resistance, circuit A must have greater current.

$\Delta V_A = \Delta V_B$, and $I_A > I_B$.

7. (6 points) The graph on the left below shows an electric potential as a function of position in one dimension. Which of the graphs i through iv shows the corresponding electric field as a function of position?

The graph begins with a constant positive slope. Since $E_x = -dV/dx$ the electric field graph must begin with a constant negative value. The graph depicting this is

Only graph iv.



8. (6 points) In the problem above, what is the direction of the electric field at the location where the electric potential is zero (that is, where the electric potential graph crosses the x axis)?

Since

$$E_x = -\frac{dV}{dx}$$

the electric field at any point is directed toward lower potential.

The electric field points in the $-x$ direction.