
I. (16 points) An infinitely long cylindrical conducting wire has radius R. A current flows along the wire with non-uniform current density $\vec{J}$. The magnitude of the current density varies with distance, r, from the cylinder axis according to

$$J = J_0 \frac{R}{r} e^{-r/R}$$

where J_0 is a positive constant.

What is the magnitude of the magnetic field at a distance $r = 2R$ from the center of the wire? Express your answer in terms of parameters defined in the problem and physical or mathematical constants.

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Use Ampere's Law. Choose a circular Amperian Loop with radius $2R$ centered on the wire.

$$\oint \vec{B} \cdot \vec{ds} = \mu_0 I_{\text{thru}} \qquad \text{and} \qquad \oint \vec{B} \cdot \vec{ds} = B 2\pi (2R) = B 4\pi R$$

Find the current from the current density. Choose the area element to be thin rings of radius r and width dr. Choose the direction of $\vec{dA}$ so it is parallel to $\vec{J}$.

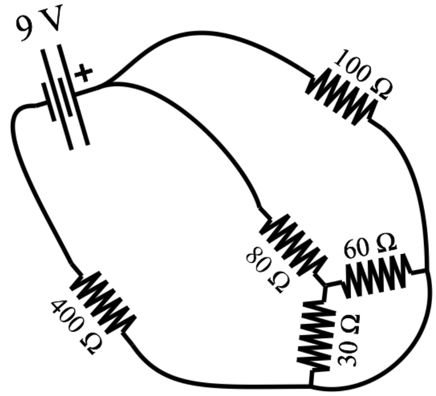
$$I_{\text{thru}} = \int \vec{J} \cdot \vec{dA} = \int_0^R J_0 \frac{R}{r} e^{-r/R} \cos 0 \, 2\pi r \, dr = 2\pi J_0 R \int_0^R e^{-r/R} \, dr = -2\pi J_0 R^2 \int_0^R e^{-r/R} \left(\frac{-1}{R} \right) dr$$

$$= -2\pi J_0 R^2 e^{-r/R} \Big|_0^R = -2\pi J_0 R^2 \left(e^{-R/R} - e^{-0/R} \right) = 2\pi J_0 R^2 (1 - 1/e)$$

So

$$\oint \vec{B} \cdot \vec{ds} = \mu_0 I_{\text{thru}} \qquad \Rightarrow \qquad B 4\pi R = \mu_0 2\pi J_0 R^2 (1 - 1/e) \qquad \Rightarrow \qquad B = \frac{\mu_0 J_0 R}{2} (1 - 1/e)$$

II. (16 points) In the circuit shown, what is the current through the $400\,\Omega$ resistor?



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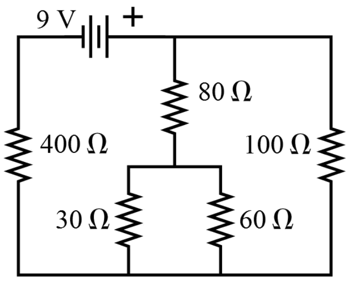
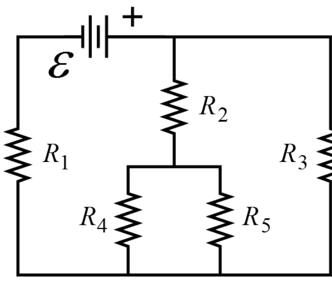
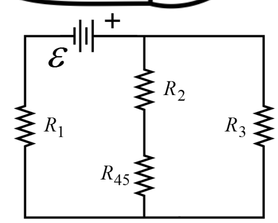
Re-draw the circuit "neater" and assign symbols to the emf and resistances.





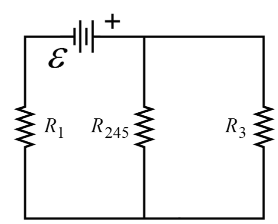


Find the equivalent resistance of the circuit. Resistors R_4 and R_5 are in parallel.

$$R_{45} = \left(\frac{1}{R_4} + \frac{1}{R_5}\right)^{-1} = \left(\frac{1}{30\,\Omega} + \frac{1}{60\,\Omega}\right)^{-1} = 20\,\Omega$$

Resistors R_2 and R_{45} are in series.

$$R_{245} = R_2 + R_{45} = 80\,\Omega + 20\,\Omega = 100\,\Omega$$

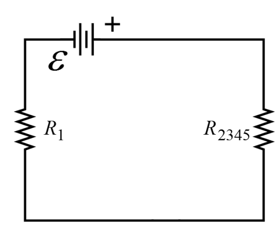


Resistors R_{245} and R_3 are in parallel.

$$R_{2345} = \left(\frac{1}{R_{245}} + \frac{1}{R_3}\right)^{-1} = \left(\frac{1}{100\,\Omega} + \frac{1}{100\,\Omega}\right)^{-1} = 50\,\Omega$$

Resistors R_1 and R_{2345} are in series.

$$R_{12345} = R_1 + R_{2345} = 400\,\Omega + 50\,\Omega = 450\,\Omega$$

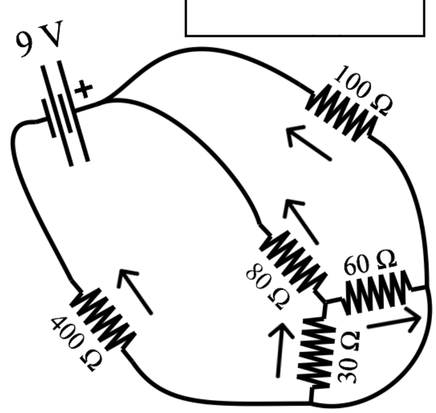


From the definition of resistance, the current supplied by the emf is

$$\Delta V = IR \quad \Rightarrow \quad I_{12345} = \frac{\Delta V_{12345}}{R_{12345}} = \frac{\mathcal{E}}{R_{12345}} = \frac{9\,\text{V}}{450\,\Omega} = 0.02\,\text{A}$$

Since resistors R_1 and R_{2345} are in series, the current through R_1, the $400\,\Omega$ resistor, is the same as the current through R_{12345}, or

$$20\,\text{mA}$$



1. (6 points) If positive current directions are defined by the arrows as shown, and current through the $400\,\Omega$ resistor is I_{400}, *etc.*, which equation is a valid expression of Kirchhoff's Loop Law?

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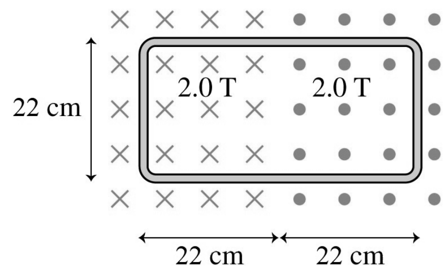
Kirchhoff's Loop Law is that the sum of potential differences around a closed loop is zero. Remember that current flows from high to low potential through a resistor. Therefore, when following such a closed loop, traversing a resistor in the direction of current flow decreases the potential, while traversing it opposite the direction of current flow increases the potential. The only offered choice consistent with this is:

$$+9\,\text{V} + I_{100}\,(100\,\Omega) + I_{60}\,(60\,\Omega) + I_{30}\,(30\,\Omega) - I_{400}\,(400\,\Omega) = 0$$

III. (16 points) A conducting rectangular loop is 22 cm high, 44 cm wide, and has a resistance of 16 Ω. Through the left half is a magnetic field of 2.0 T into the page, while through the right side is a magnetic field of 2.0 T out of the page, as shown. At time $t = 0$ the field magnitude into the left half begins to decrease at 0.50 T/s, while the magnitude out of the right half begins to increase as 0.50 T/s. What is the current in the loop 1.0 s later? If the current must be zero, prove it.



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Use Faraday's Law to find the emf induced in the loop.

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = \frac{-d}{dt}\int \vec{B}\cdot d\vec{A} = \frac{-d}{dt}\int B\cos\theta\, dA$$

As current direction is not asked, only magnitudes need be considered. Although the magnetic field depends on position, the emf depends only the change in magnetic field. The field and area vectors are either parallel or anti-parallel (that is, $\cos\theta$ is either 1 or -1).

$$|\mathcal{E}| = \left|\frac{d}{dt}\int B\, dA\right| = \left|\frac{d\,(BA)}{dt}\right| = A\left|\frac{dB}{dt}\right| = (0.22\,\text{m})\,(0.44\,\text{m})\,(0.50\,\text{T/s}) = 0.0484\,\text{V}$$

The current can be found using the definition of resistance.

$$I = \frac{\mathcal{E}}{R} = \frac{0.0484\,\text{V}}{16\,\Omega} = 0.0030\,\text{A} = 3.0\,\text{mA}$$

2. (6 points) In the problem above, what direction does current flow in the loop at time $t = 1.0\,\text{s}$?

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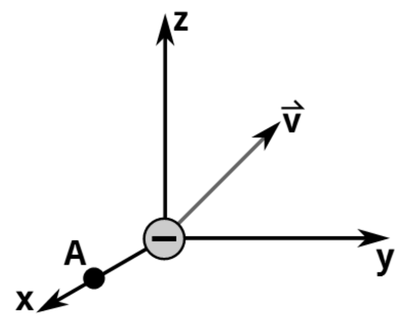
With the magnetic field into the left half decreasing and that out of the right half increasing, the magnetic flux is becoming more outward everywhere in the loop. Lenz' Law requires that the current flow to oppose this change. Therefore, the current will flow to make a magnetic field into the page in the center of the loop. By the short-cut Right-Hand-Rule, this current is

Clockwise

3. (8 points) A negatively-charged particle is moving in the yz-plane. At the moment when it passes through the origin, its velocity is given by

$$\vec{v} = \frac{1}{\sqrt{2}}\hat{\jmath} + \frac{1}{\sqrt{2}}\hat{k}$$



If a, b, and c are positive constants, what is the direction of the magnetic field at point A on the positive x-axis?

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Use the Biot-Savart Law.

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{q\vec{v}\times\hat{r}}{|\vec{r}|^2}$$

With $\hat{r}$ in the $+x$ direction, the Right-Hand-Rule shows us that $\vec{v}\times\hat{r}$ is down and to the right on the page (positive y and negative z). Since the particle is negative, $q\vec{v}\times\hat{r}$ must be in the opposite direction, up and to the left on the page (negative y and positive z). That direction is expressed by

$$-b\hat{\jmath} + c\hat{k}$$

4. (8 points) A long straight wire carries current I in the $+z$ direction. A nearby square loop lies in the $x-y$ plane and carries a current I that is counter-clockwise when viewed from a position on the $+z$ axis, as shown. How, if at all, will the loop rotate if released from this position?

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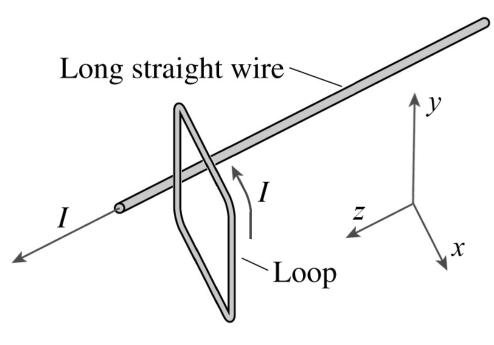

Using the short-cut Right-Hand-Rule for the magnetic field due to a long straight wire, the field is in the $+y$ direction at the location of the loop. Using the short-cut Right-Hand-Rule for the magnetic field in the center of a loop, the loop's magnetic field, and thus its magnetic moment, is in the $+z$ direction. The torque on a current loop

$$\vec{\tau} = \vec{\mu}\times\vec{B}$$

tends to align the magnetic moment with the field. So ...

It will rotate clockwise as viewed from the $+x$ axis.

5. (8 points) A circuit is constructed with a 48 V battery, a switch, a 30 μF capacitor, an 8 Ω and a 16 Ω resistor, as shown. What is the current through the 16 Ω resistor after the switch has been closed for a long time?

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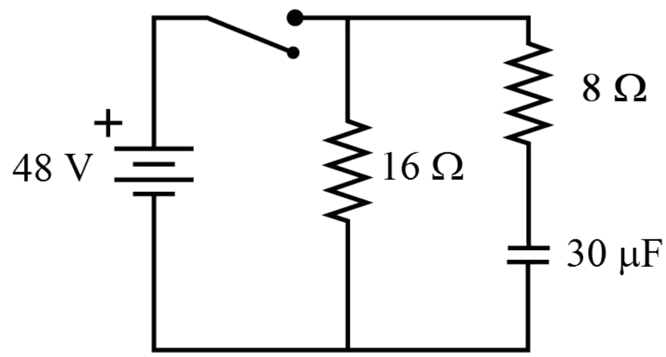

Regardless of how long the switch has been closed, there is always 48 V across the 16 Ω resistor. From the definition of resistance,

$$I = \frac{\Delta V_R}{R} = \frac{48\,\text{V}}{16\,\Omega} = 3.0\,\text{A}$$

6. (8 points) A circuit is constructed with a 48 V battery, a switch, a 30 mH inductor, an 8 Ω and a 16 Ω resistor, as shown. The switch has been closed for a long time. What is the current through the 16 Ω resistor immediately upon opening the switch?

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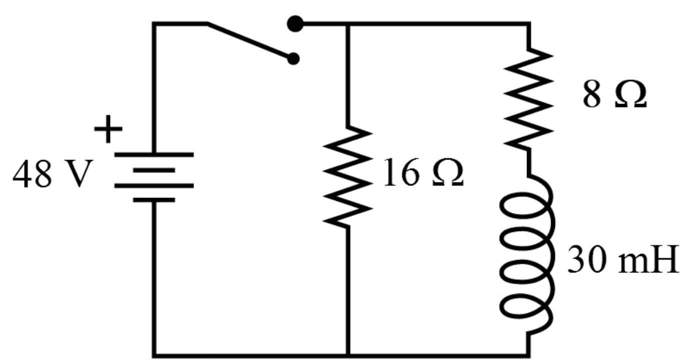

When the switch has been closed a long time, there are no more changes in current, so there is no emf induced in the inductor. There is 48 V across the 8 Ω resistor. From the definition of resistance there are

$$I = \frac{\Delta V_R}{R} = \frac{48\,\text{V}}{8\,\Omega} = 6.0\,\text{A}$$

through it. As the inductor is in series with the 8 Ω resistor, the current through the inductor must also be 6.0 A. When the switch is opened, the inductor opposes any change in current, and maintains 6.0 A for an instant. That current must pass through the 16 Ω resistor.

6.0 A

7. (8 points) The circuit shown to the right contains three light bulbs rated at 100 W. Compare the brightnesses of the bulbs after the switch has been closed to their original brightnesses, when the switch is open. *Hint:* Consider the potentials across the bulbs.

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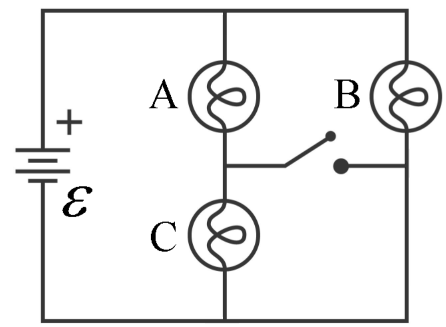

While the switch is open, bulbs A and C each have potential difference $\mathcal{E}/2$ across them, and bulb B has potential difference $\mathcal{E}$. After the switch is closed, bulbs A and B each have potential difference $\mathcal{E}$ across them, and bulb C has no potential difference across it (it is shorted out). Since the current through the bulbs is proportional to the potential difference, and the current determines the brightness,

Bulb A becomes brighter, bulb B remains at the same brightness, and bulb C becomes dimmer.

Quiz and Exam Formulæ & Constants

$$k = \frac{1}{4\pi\epsilon_0}$$
$$\Delta V = -\int \vec{E}\cdot d\vec{s}$$
$$V = k\frac{q}{r}$$
$$\Delta U = q\,\Delta V$$
$$I = dq/dt$$
$$P = I\,\Delta V$$
$$R = \frac{\Delta V}{I}$$

Series :
$$\frac{1}{C_{\text{eq}}} = \sum \frac{1}{C_i}$$
$$R_{\text{eq}} = \sum R_i$$

Parallel :
$$\frac{1}{R_{\text{eq}}} = \sum \frac{1}{R_i}$$
$$C_{\text{eq}} = \sum C_i$$

$$\vec{E} = k\frac{q}{r^2}\hat{r}$$
$$\vec{F} = k\frac{q_1 q_2}{r^2}\hat{r}$$
$$\vec{F} = q\vec{E}$$
$$\vec{p} = q\vec{d}$$
$$\vec{\tau} = \vec{p}\times\vec{E}$$
$$U = -\vec{p}\cdot\vec{E}$$
$$|\vec{E}| \propto \frac{|\vec{p}|}{r^3}$$
$$\Phi_{\text{E}} = \int \vec{E}\cdot d\vec{A}$$
$$\epsilon_0 \oint \vec{E}\cdot d\vec{A} = q_{\text{enclosed}}$$
$$\oint \vec{E}\cdot d\vec{\ell} = -\frac{d\Phi_{\text{B}}}{dt}$$
$$C = \frac{Q}{\Delta V}$$
$$C = \epsilon_0 \frac{A}{d}$$
$$U = \tfrac{1}{2}C\,[\Delta V]^2$$
$$R = \rho\frac{\ell}{A}$$
$$\tau_{\text{C}} = RC$$
$$u_{\text{E}} = \tfrac{1}{2}\epsilon_0 E^2$$

$$\vec{B} = \frac{\mu_0 q}{4\pi}\,\frac{\vec{v}\times\hat{r}}{r^2}$$
$$d\vec{B} = \frac{\mu_0 I}{4\pi}\,\frac{d\vec{\ell}\times\hat{r}}{r^2}$$
$$\vec{F} = q\vec{v}\times\vec{B}$$
$$\vec{F} = I\vec{\ell}\times\vec{B}$$
$$\vec{\mu} = NI\vec{A}$$
$$\vec{\tau} = \vec{\mu}\times\vec{B}$$
$$U = -\vec{\mu}\cdot\vec{B}$$
$$\Phi_{\text{B}} = \int \vec{B}\cdot d\vec{A}$$
$$\oint \vec{B}\cdot d\vec{A} = 0$$
$$\oint \vec{B}\cdot d\vec{\ell} = \mu_0(I_{\text{c}} + I_{\text{d}})$$
$$L = \frac{\Phi_{\text{B}}}{I}$$
$$L = \mu_0 N^2 \frac{A}{\ell}$$
$$U = \tfrac{1}{2}LI^2$$
$$B = \mu_0 n I$$
$$\tau_{\text{L}} = L/R$$
$$u_{\text{B}} = \frac{1}{2\mu_0}B^2$$

$$q = q_{\max}\left(1 - e^{-t/\tau_{\text{C}}}\right)$$
$$q = q_0 e^{-t/\tau_{\text{C}}}$$
$$I = I_{\max}\left(1 - e^{-t/\tau_{\text{L}}}\right)$$
$$I = I_0 e^{-t/\tau_{\text{L}}}$$
$$I = \int \vec{J}\cdot d\vec{A}$$
$$\vec{J} = \sigma\vec{E}$$
$$\mathcal{E} = -N\frac{d\Phi_{\text{B}}}{dt}$$
$$I_d = \epsilon_0 \frac{d\Phi_{\text{E}}}{dt}$$
$$\mathcal{E} = -L\frac{dI}{dt}$$
$$c = f\lambda = \frac{|\vec{E}|}{|\vec{B}|}$$
$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$$

Fundamental Charge $e = 1.602\times10^{-19}\,\text{C}$
Earth's gravitational field $g = 9.81\,\text{N/kg}$
Coulomb constant $K = 8.988\times10^{9}\,\text{N}\cdot\text{m}^2/\text{C}^2$
Speed of Light $c = 2.998\times10^{8}\,\text{m/s}$

Mass of an Electron $m_{\text{e}} = 9.109\times10^{-31}\,\text{kg}$
Mass of a Proton $m_{\text{p}} = 1.673\times10^{-27}\,\text{kg}$
Vacuum Permittivity $\epsilon_0 = 8.854\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2$
Vacuum Permeability $\mu_0 = 4\pi\times10^{-7}\,\text{T}\cdot\text{m/A}$

Unless otherwise directed, friction, drag, and gravity should be neglected, and all batteries and wires are ideal.
All derivatives and integrals in free-response problems must be evaluated.

You may remove this sheet from your Quiz or Exam