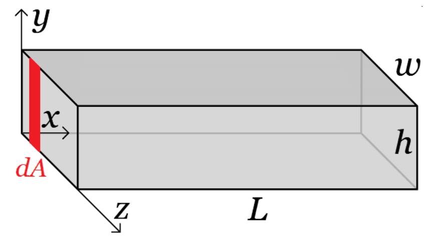

I. (16 points) A wire of length L has rectangular cross-section h high and w wide. There is a uniform electric field of magnitude E_0 within it, in the x direction.

$$\vec{E} = E_0\,\hat{\imath}$$

The conductivity of the wire is non-uniform, however, and depends on position z according to

$$\sigma = \sigma_0\,\frac{z}{w}$$



where σ_0 is a positive constant. In terms of parameters defined in the problem, and physical or mathematical constants, what is the current in the wire?

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Current can be found from current density. Current density is related to electric field.

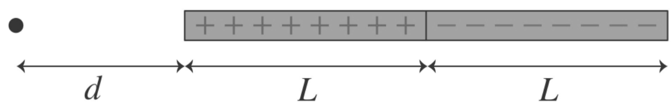
$$I = \int \vec{J}\cdot d\vec{A} \qquad \text{and} \qquad \vec{J} = \sigma\vec{E} = \sigma_0\,\frac{z}{w}\,E_0\,\hat{\imath}$$

Choose $d\vec{A}$ parallel to $\vec{E}$. That is, the perpendicular to the surface dA should point in the same direction as the electric field. In this case, that is the $+x$ direction. As the conductivity (and thus the current density) varies with z, $d\vec{A}$ should be "small" in the z direction. Therefore, $d\vec{A} = h\,dz\,\hat{\imath}$, as shown. Then, remembering that $\hat{\imath}\cdot\hat{\imath} = 1$,

$$I = \int \vec{J}\cdot d\vec{A} = \int_0^w \left(\sigma_0\,\frac{z}{w}\,E_0\,\hat{\imath}\right)\cdot(h\,dz\,\hat{\imath}) = \int_0^w \sigma_0\,\frac{z}{w}\,E_0\,h\,dz$$

$$= \frac{\sigma_0 E_0 h}{w}\int_0^w z\,dz = \frac{\sigma_0 E_0 h}{w}\left[\frac{z^2}{2}\right]_0^w = \frac{\sigma_0 E_0 h}{2w}\left(w^2 - 0\right) = \sigma_0 E_0 h w/2$$

II. (16 points) The two halves of a rod, each of length L, are uniformly charged to $\pm Q$, with the left half positive and the right half negative, as shown. What is the electric potential (with respect to zero at infinity) at the point indicated by the dot, a distance d from the positively-charged end of the rod? Express your answer in terms of parameters defined in the problem, and physical or mathematical constants. If the potential is necessarily zero, prove it.



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Chose a coordinate system. I'll chose the origin at the dot, and positive x to the right.

Each infinitesimal element, dx, of the rod has charge dq. Because the element is pointlike, it contributes potential dV at the origin as if it were a point charge:

$$dV = \frac{K\,dq}{r}$$

Adding up the contributions of all the elements is an integration.

$$V = \int dV = \int \frac{K\,dq}{r} = \int \frac{K\lambda\,dx}{x} = \int_{d}^{d+L} \frac{K\,(+Q/L)\;dx}{x} + \int_{d+L}^{d+2L} \frac{K\,(-Q/L)\;dx}{x}$$

$$= \frac{KQ}{L}\ln x\bigg|_{d}^{d+L} - \frac{KQ}{L}\ln x\bigg|_{d+L}^{d+2L} = \frac{KQ}{L}\left[\ln\frac{d+L}{d} - \ln\frac{d+2L}{d+L}\right]$$

$$= \frac{KQ}{L}\ln\frac{(d+L)^2}{d\,(d+2L)}$$

1. (6 points) In the problem above, what is the direction of the electric potential at the dot?

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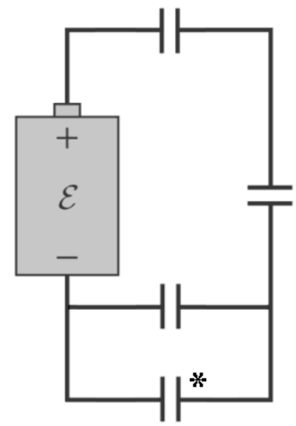
Electric potential is a scalar!

This is not a meaningful question.

III. (16 points) The battery in the illustrated circuit has emf $\mathcal{E}$. Each of the four capacitors has the same capacitance, C. Once the circuit has been connected for a long time, what energy is stored in the bottom-most capacitor, marked with an asterisk, with respect to zero energy at zero charge? Express your answer in terms of parameters defined in the problem, and physical or mathematical constants.



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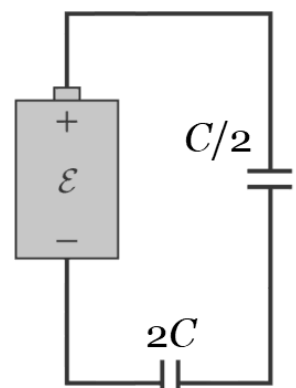
The top and right-most capacitors are in series (same charge). Their equivalent capacitance can be found.

$$\frac{1}{C_{\text{eq1}}} = \frac{1}{C} + \frac{1}{C} \qquad \Rightarrow \qquad C_{\text{eq1}} = \frac{C}{2}$$

The two bottom capacitors are in parallel (same potential difference). Their equivalent capacitance can be found.

$$C_{\text{eq2}} = C + C = 2C$$



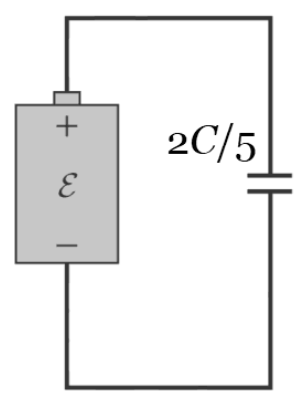
These two new equivalent capacitances are in series. The equivalent capacitance for the entire circuit can be found.

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C/2} + \frac{1}{2C} \qquad \Rightarrow \qquad C_{\text{eq}} = \frac{2C}{5}$$

The charge delivered by the battery is

$$Q = C_{\text{eq}}\,\Delta V = \left(\frac{2C}{5}\right)\mathcal{E}$$

This charge is on the $2C/5$ equivalent capacitor, and on each of the $C/2$ and $2C$ equivalent capacitors (since they are in series). The potential difference across the $2C$ equivalent capacitor is



$$\Delta V = \frac{Q}{C_{\text{eq2}}} = \frac{(2C/5)\,\mathcal{E}}{2C} = \frac{\mathcal{E}}{5}$$

Since the two bottom capacitors are in parallel, this is also the potential difference across each of them individually. The energy stored in the bottom-most capacitor, then, is

$$U = \tfrac{1}{2}C\,(\Delta V)^2 = \tfrac{1}{2}C\left(\frac{\mathcal{E}}{5}\right)^2 = \frac{C\mathcal{E}^2}{50}$$

2. (6 points) If the energy found in the problem above is U_0, what energy would be stored in that same capacitor if the battery were replaced by one with twice the emf, $2\mathcal{E}$?

When the battery has emf $2\mathcal{E}$, the energy stored in the bottom-most capacitor is ...

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The stored energy is proportional to the square of the potential across the capacitor. In a circuit with only one battery, the energy stored in each capacitor must be proportional to the square of the battery's emf.

The energy stored in the bottom-most capacitor is $4U_0$.

3. (8 points) Two positive charges are located on the x axis at the points $(-a, 0)$ and $(a, 0)$; two more charges of the same magnitude, but opposite sign, are located on the y axis at the points $(0, -a)$ and $(0, a)$. Rank the electric potentials V_i at the points $P_1 = (0, 0)$, $P_2 = (-2a, 0)$, $P_3 = (-2a, -2a)$, $P_4 = (2a, 2a)$, and $P_5 = (0, 2a)$.

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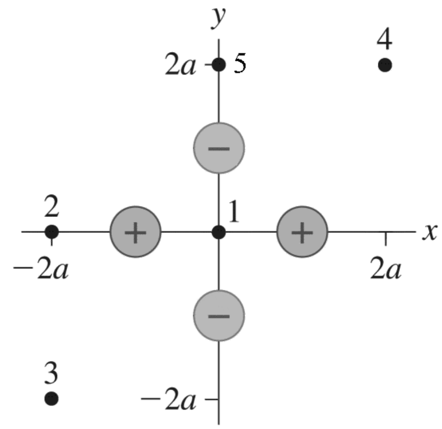

Points 1, 3, and 4 are all equidistant from the same amount of positive and negative charge. The potentials at those points are zero, with respect to zero at infinity.

Point 2 is a positive potential, as

$$K\frac{q}{a} + 2K\frac{-q}{a\sqrt{5}} + K\frac{q}{3a} = \frac{Kq}{a}\left(1 - \frac{2}{\sqrt{5}} + \frac{1}{3}\right) > 0$$

Similarly, point 5 is at negative potential, so

$$V_2 > V_1 = V_3 = V_4 > V_5$$

4. (8 points) The illustrated wire is composed of two segments, 1 and 2, having different conductivities σ_1 and σ_2, and different radii R_1 and R_2. The conductivity of segment 2 is only half the conductivity of segment 1. The radius of segment 2 is three times the radius of segment 1. What is the ratio of the electric field magnitude in segment 1 to that in segment 2, E_1/E_2?

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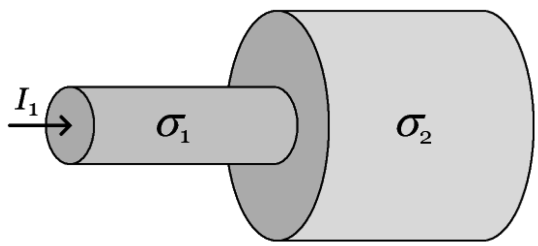

The electric field can be related to the current density and the current. In terms of magnitudes

$$J = \sigma E = \frac{I}{A}$$

The current in each segment is the same.

$$I_1 = I_2 \qquad \Rightarrow \qquad \sigma_1 E_1 A_1 = \sigma_2 E_2 A_2$$

So

$$\frac{E_1}{E_2} = \frac{\sigma_2 A_2}{\sigma_1 A_1} = \frac{\sigma_2 \pi r_2^2}{\sigma_1 \pi r_1^2} = \frac{\sigma_2}{2\sigma_2}\left(\frac{3r_1}{r_1}\right)^2 = \frac{9}{2}$$

5. (8 points) Electric potential is graphed as a function of position. Where is the electric field magnitude greatest? At that location, in which direction does the electric field point?

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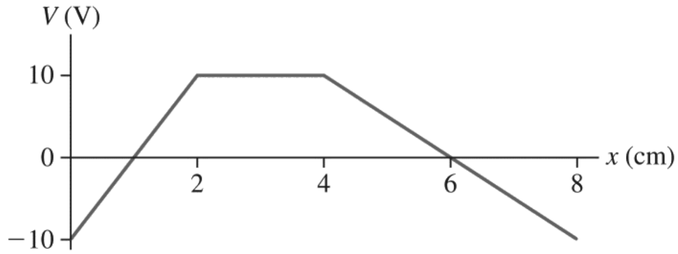
Since the electric field is related to the electric potential by

$$\vec{E} = -\frac{\delta V}{\delta s}$$

the greatest field magnitude will be where the slope of the graph has greatest magnitude, and the electric field points toward lower potential.



Electric field has greatest magnitude between 0 and 2 cm. It points in the negative direction.

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6. (8 points) A parallel plate capacitor has been connected to a battery for a long time. Insulating handles will be attached to the plates, and the plates will be pushed closer together. How does the energy stored in the capacitor change, depending on whether or not the battery is disconnected before moving the plates?

. . . .

Remember the expression for the capacitance of parallel plates

$$C = \epsilon_0 \frac{A}{d}$$

When the plates are pushed closer together, d, the distance between them, decreases, so the capacitance increases.

Now consider the two (related) expressions for the potential energy stored in a capacitor

$$U = \frac{1}{2} C (\Delta V)^2 \qquad \text{and} \qquad U = \frac{Q^2}{2C}$$

When the plates are left connected to the battery, the electric potential difference between them is constant, so when the capacitance decreases then the stored energy decreases. However, when the plates are disconnected from the battery, the charge on them is constant, so when the capacitance decreases then the stored energy increases.

If the plates remain connected the stored energy increases,
but if they are disconnected it decreases.

7. (8 points) A system consists of three charged particles at the vertices of an equilateral triangle, as illustrated. Two of the particles have charge of equal magnitude, q, but opposite sign. The third particle has charge Q. How does the electric potential energy of the system change if the particle with charge Q is removed to infinitely far away?

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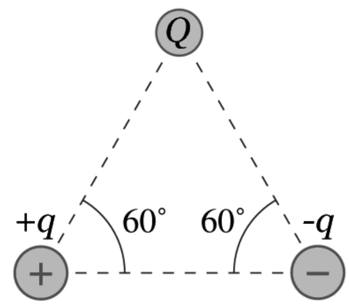

The particle with charge Q is equidistant from charged particles of equal magnitude but opposite sign. Therefore, it is at a position with an electric potential of zero, with respect to zero at infinite distance. So, when it is removed to infinitely far away, there is no change in its electric potential, and no change in the electric potential energy of the system.

It remains the same.

Quiz and Exam Formulæ & Constants

$$k = \frac{1}{4\pi\epsilon_0}$$
$$\Delta V = -\int \vec{E}\cdot d\vec{s}$$
$$V = k\frac{q}{r}$$
$$\Delta U = q\,\Delta V$$
$$I = dq/dt$$
$$P = I\,\Delta V$$
$$R = \frac{\Delta V}{I}$$

Series :
$$\frac{1}{C_{\text{eq}}} = \sum \frac{1}{C_i}$$
$$R_{\text{eq}} = \sum R_i$$

Parallel :
$$\frac{1}{R_{\text{eq}}} = \sum \frac{1}{R_i}$$
$$C_{\text{eq}} = \sum C_i$$

$$\vec{E} = k\frac{q}{r^2}\hat{r}$$
$$\vec{F} = k\frac{q_1 q_2}{r^2}\hat{r}$$
$$\vec{F} = q\vec{E}$$
$$\vec{p} = q\vec{d}$$
$$\vec{\tau} = \vec{p}\times\vec{E}$$
$$U = -\vec{p}\cdot\vec{E}$$
$$|\vec{E}| \propto \frac{|\vec{p}|}{r^3}$$
$$\Phi_{\text{E}} = \int \vec{E}\cdot d\vec{A}$$
$$\epsilon_0 \oint \vec{E}\cdot d\vec{A} = q_{\text{enclosed}}$$
$$\oint \vec{E}\cdot d\vec{\ell} = -\frac{d\Phi_{\text{B}}}{dt}$$
$$C = \frac{Q}{\Delta V}$$
$$C = \epsilon_0\frac{A}{d}$$
$$U = \tfrac{1}{2}C\,[\Delta V]^2$$
$$R = \rho\frac{\ell}{A}$$
$$\tau_{\text{C}} = RC$$
$$u_{\text{E}} = \tfrac{1}{2}\epsilon_0 E^2$$

$$\vec{B} = \frac{\mu_0 q}{4\pi}\,\frac{\vec{v}\times\hat{r}}{r^2}$$
$$d\vec{B} = \frac{\mu_0 I}{4\pi}\,\frac{d\vec{\ell}\times\hat{r}}{r^2}$$
$$\vec{F} = q\vec{v}\times\vec{B}$$
$$\vec{F} = I\vec{\ell}\times\vec{B}$$
$$\vec{\mu} = NI\vec{A}$$
$$\vec{\tau} = \vec{\mu}\times\vec{B}$$
$$U = -\vec{\mu}\cdot\vec{B}$$
$$\Phi_{\text{B}} = \int \vec{B}\cdot d\vec{A}$$
$$\oint \vec{B}\cdot d\vec{A} = 0$$
$$\oint \vec{B}\cdot d\vec{\ell} = \mu_0(I_{\text{c}} + I_{\text{d}})$$
$$L = \frac{\Phi_{\text{B}}}{I}$$
$$L = \mu_0 N^2\frac{A}{\ell}$$
$$U = \tfrac{1}{2}LI^2$$
$$B = \mu_0 n I$$
$$\tau_{\text{L}} = L/R$$
$$u_{\text{B}} = \frac{1}{2\mu_0}B^2$$

$$q = q_{\max}\left(1 - e^{-t/\tau_{\text{C}}}\right)$$
$$q = q_0 e^{-t/\tau_{\text{C}}}$$
$$I = I_{\max}\left(1 - e^{-t/\tau_{\text{L}}}\right)$$
$$I = I_0 e^{-t/\tau_{\text{L}}}$$
$$I = \int \vec{J}\cdot d\vec{A}$$
$$\vec{J} = \sigma\vec{E}$$
$$\mathcal{E} = -N\frac{d\Phi_{\text{B}}}{dt}$$
$$I_d = \epsilon_0\frac{d\Phi_{\text{E}}}{dt}$$
$$\mathcal{E} = -L\frac{dI}{dt}$$
$$c = f\lambda = \frac{|\vec{E}|}{|\vec{B}|}$$
$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$$

Fundamental Charge $e = 1.602\times10^{-19}\,\text{C}$
Earth's gravitational field $g = 9.81\,\text{N/kg}$
Coulomb constant $K = 8.988\times10^{9}\,\text{N}\cdot\text{m}^2/\text{C}^2$
Speed of Light $c = 2.998\times10^{8}\,\text{m/s}$

Mass of an Electron $m_{\text{e}} = 9.109\times10^{-31}\,\text{kg}$
Mass of a Proton $m_{\text{p}} = 1.673\times10^{-27}\,\text{kg}$
Vacuum Permittivity $\epsilon_0 = 8.854\times10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2$
Vacuum Permeability $\mu_0 = 4\pi\times10^{-7}\,\text{T}\cdot\text{m/A}$

Unless otherwise directed, friction, drag, and gravity should be neglected, and all batteries and wires are ideal.
All derivatives and integrals in free-response problems must be evaluated.

You may remove this sheet from your Quiz or Exam