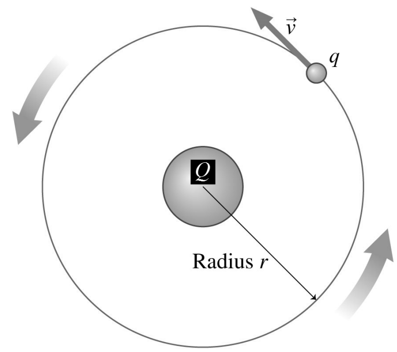

I. (16 points) A uniform sphere with charge $Q = 6.0\,\mu\text{C}$ is fixed in place. A particle with mass $m = 2.5\,\text{mg}$ and tangential speed $v = 212\,\text{m/s}$ circles the sphere in an orbit with radius $r = 1.4\,\text{m}$. What is the charge q of the particle?



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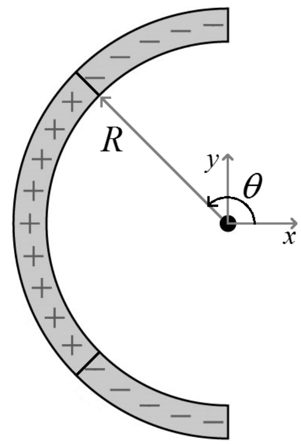
Use Newton's Second Law. Choose a coordinate system that points in the known direction of the acceleration, which is toward the center of the circular orbit. The only force on the particle is the electric force, found using Coulomb's Law.

$$\sum F_c = F_E = ma_c \qquad \Rightarrow \qquad K\frac{Qq}{r^2} = m\frac{v^2}{r}$$

So, remembering to convert the particle's mass to kilograms,

$$q = \frac{mv^2 r}{KQ} = \frac{\left(2.5\times10^{-6}\,\text{kg}\right)\left(212\,\text{m/s}\right)^2\left(1.4\,\text{m}\right)}{\left(8.988\times10^{9}\,\text{N·m}^2/\text{C}^2\right)\left(6.0\times10^{-6}\,\text{C}\right)} = 2.9\,\mu\text{C}$$

1. (6 points) A plastic rod is bent into a semi-circle of radius R, as shown. It has a uniform linear charge density **magnitude** λ, but the charge density is negative from $\theta = \pi/2$ to $3\pi/4$, positive from $\theta = 3\pi/4$ to $5\pi/4$, then negative again from $\theta = 5\pi/4$ to $3\pi/2$. In what direction, if any, is the electric field at the center of the arc?



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The negative quarters of the arc would attract a positive probe charge placed at the origin, while the positive quarters would repel it. The y components of this force due to each pair of quarters will cancel. Each negative quarter produces exerts a force with more y component and less x component than the positive quarters do. The direction of the net force on a positive probe charge, and thus the direction of the electric field, is

$+x$

II. (16 points) In the problem above, find the magnitude of the electric field at the center of the arc, in terms of parameters defined in the problem and physical or mathematical constants. If the field magnitude is necessarily zero, prove it.

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In general, a segment of arc produces field with x component

$$E_x = \int dE_x = \int dE\,\cos\theta = \int \frac{K\,dq}{r^2}\cos\theta = \int \frac{K\lambda\,ds}{R^2}\cos\theta = \int_{\theta_1}^{\theta_2} \frac{K\lambda R\,d\theta}{R^2}\cos\theta = \frac{K\lambda}{R}\int_{\theta_1}^{\theta_2}\cos\theta\,d\theta$$

where an infinitesimal bit of arc length $ds = R\,d\theta$ has an infinitesimal bit of charge $dq = \lambda\,ds$.

Consider the segments on the $+y$ side of the x axis. The negative segment produces field with a negative x component

$$E_{-x} = \frac{K\lambda}{R}\int_{\pi/2}^{3\pi/4}\cos\theta\,d\theta = \frac{K\lambda}{R}\sin\theta\Big|_{\pi/2}^{3\pi/4} = \frac{K\lambda}{R}\left[\sin\left(\frac{3\pi}{4}\right) - \sin\left(\frac{\pi}{2}\right)\right] = \frac{K\lambda}{R}\left[\frac{\sqrt{2}}{2} - 1\right]$$

which is negative, as it should be. The positive segment produces field with a positive x component

$$E_{+x} = \frac{K\lambda}{R}\int_{3\pi/4}^{\pi}\cos\theta\,d\theta = \frac{K\lambda}{R}\sin\theta\Big|_{3\pi/4}^{\pi} = \frac{K\lambda}{R}\left[\sin(\pi) - \sin\left(\frac{3\pi}{4}\right)\right] = \frac{K\lambda}{R}\left[0 - \frac{\sqrt{2}}{2}\right] \quad\Rightarrow\quad \frac{K\lambda}{R}\left[\frac{\sqrt{2}}{2}\right]$$

where the sign has been changed to make this component positive.

The x component of the field in the center due to the top half of the arc, then, is

$$E_{\text{top}} = E_{+x} + E_{-x} = \frac{K\lambda}{R}\left[\frac{\sqrt{2}}{2}\right] + \frac{K\lambda}{R}\left[\frac{\sqrt{2}}{2} - 1\right] = \frac{K\lambda}{R}\left[\sqrt{2} - 1\right]$$

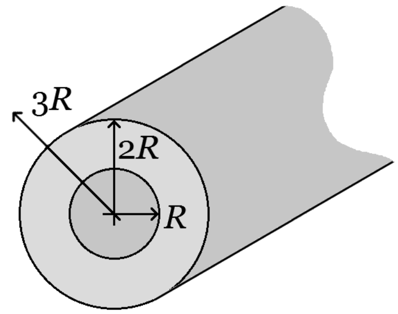
The bottom half of the arc contributes an equal amount.

$$E_{\text{total}} = 2E_{\text{top}} = \frac{2K\lambda}{R}\left[\sqrt{2} - 1\right]$$

III. (16 points) An infinite hollow insulating cylinder has inner radius R and outer radius $2R$, as illustrated. Its volume charge density, ρ, varies with distance r from the center according to

$$\rho = \rho_0 \left(\frac{R}{r}\right)$$

where ρ_0 is a positive constant. Find the electric field magnitude at a distance $3R$ from the center in terms of parameters defined in the problem, and physical or mathematical constants.



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Use Gauss' Law, $\epsilon_0 \Phi_E = \epsilon_0 \oint \vec{E} \cdot d\vec{A} = q_{in}$. Choose a surface that passes through the point at which the electric field is to be found, and with the same symmetry as the charge distribution. A finite cylinder of radius $3R$ and length ℓ satisfies these conditions.

Note that the flux through the ends of the Gaussian Surface is zero, as the electric field vectors are perpendicular to the outward-pointing area vectors. The electric field has constant magnitude over the curved side of the Gaussian Surface, and is parallel to the outward-pointing area vectors.

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \oint E \cos\theta \, dA = E \cos(0°) \oint dA = EA = E2\pi r\ell = E2\pi \, 3R \, \ell = E6\pi R\ell$$

The charge inside the Gaussian Surface can be found from the volume charge density, as $\rho = dq/dV$. Choose a thin cylindrical shell for the volume element, $dV = 2\pi r\ell \, dr$.

$$q_{in} = \int \rho \, dV = \int_R^{2R} \rho_0 \left(\frac{R}{r}\right) 2\pi r\ell \, dr = \rho_0 R 2\pi\ell \int_R^{2R} dr = \rho_0 R 2\pi\ell r \Big|_R^{2R}$$

$$= \rho_0 R 2\pi\ell \Big[2R - R\Big] = \rho_0 R^2 2\pi\ell$$

So

$$\epsilon_0 \Phi_E = q_{in} \qquad \Rightarrow \qquad \epsilon_0 E6\pi R\ell = \rho_0 R^2 2\pi\ell \qquad \Rightarrow \qquad E = \frac{\rho_0 R}{3\epsilon_0}$$

2. (6 points) In the problem above, let the magnitude of the electric field at a distance $3R$ from the center be E_0. What is the magnitude of the electric field at distance $R/3$ from the center?

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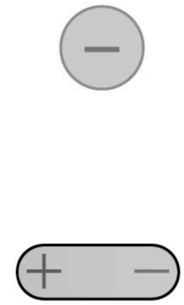
A Gaussian Surface with the symmetry of the charge distribution (a cylinder) that passes through a point $R/3$ from the center would contain no charge. Therefore, there would be no flux through that surface, and the field at that point would be

Zero

3. (8 points) An electric dipole is released from rest near negatively-charged particle that is fixed in place, as shown. What is the subsequent motion, if any, of the dipole?



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The electric dipole with align with the field lines due to the point charge, and move toward greater field strength.

The dipole rotates clockwise and moves up the page.

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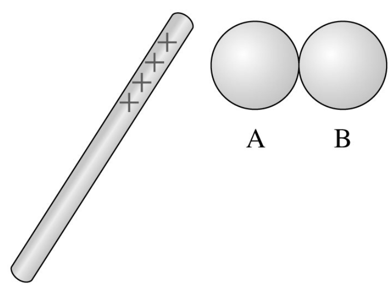
4. (8 points) Identical conducting spheres A and B are uncharged and touching. A positively-charged rod is brought near sphere A, as shown. The spheres are separated, and the rod is removed. Describe the charge on each sphere.



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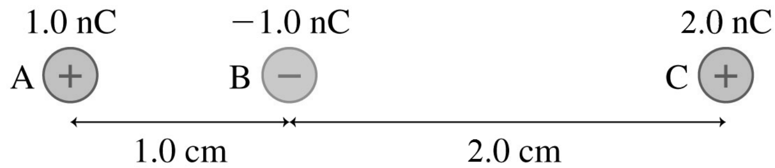
While the spheres are touching, they will act as a single conductor, becoming polarized by the rod. Negative charge will be attracted toward the rod into sphere A, leaving positive charge behind on sphere B. Separating the spheres traps that charge, so it cannot recombine when the rod is removed.

A is negative, B is positive.

5. (8 points) Three charged particles are arranged in a line, as shown. What is the magnitude of the net force on particle B?



Use Coulomb's Law. There will be an attractive force between particles A and B with magnitude

$$F_{AB} = K\frac{Q_A Q_B}{R_{AB}^2}$$

$$= \left(9.0 \times 10^9\,\mathrm{N{\cdot}m^2/C^2}\right)\frac{\left(1.0 \times 10^{-9}\,\mathrm{C}\right)(1.0\,\mathrm{nC})}{\left(1.0 \times 10^{-2}\,\mathrm{m}\right)^2} = 9.0 \times 10^4\,\mathrm{nN}$$

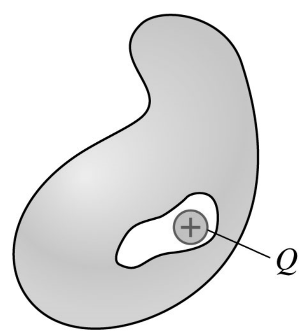
Particle C has twice the charge of particle A, but is twice as far from B as A is. Therefore, the force between C and B will be half the force between A and B.

$$F_{AB} = F_{AB}/2 = \frac{9.0 \times 10^4\,\mathrm{nN}}{2} = 4.5 \times 10^4\,\mathrm{nN}$$

This is also an attractive force, so the net force magnitude on B is the difference between the force magnitudes from A and C

$$F_B = F_{AB} - F_{CB} = 9.0 \times 10^4\,\mathrm{nN} - 4.5 \times 10^4\,\mathrm{nN} = 4.5 \times 10^4\,\mathrm{nN}$$

6. (8 points) A conducting object contains a hollow void. Within that void lies a particle with positive charge $+Q$, as shown. What net charge must the conducting object have if the charge on the surface of the void is twice the charge on the outer surface?



As there can be no electric field in the bulk of a conductor at equilibrium, a Gaussian Surface that contains the void will have no net flux, and thus no net charge inside it. Therefore, there must be a charge $-Q$ on the surface of the void. If that's twice the charge on the outer surface, the outer surface must have a charge $-Q/2$, resulting in a total charge on the conductor of

$$-3Q/2$$

7. (8 points) An equilateral prism has length L. The triangular faces have sides of length s. A uniform electric field with magnitude E_0 is directed up the page, perpendicular to the bottom face of the prism. What is the magnitude of the electric flux through the shaded face?

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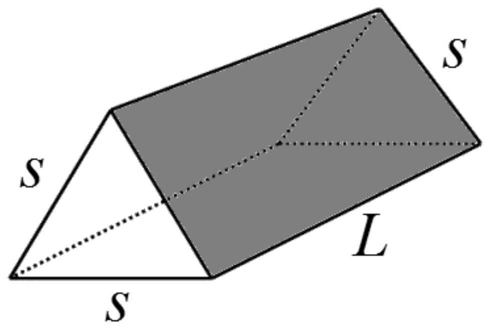

As the field is uniform and perpendicular to the bottom surface, there is a flux magnitude

$$\Phi = EA = E_0 s L$$

at that surface. Half the field lines entering the bottom surface exit through each of the upper surfaces. The flux magnitude through one upper surface, then, must be

$$E_0 s L/2$$