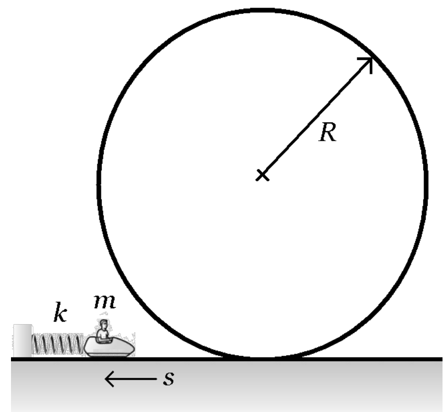

I. (16 points) An amusement park sled and rider, with total mass m, is to be sent around a loop-the-loop of radius R. The sled and rider will accomplish this by compressing a large spring with Hooke's Law constant k, and being released from rest. The ground and loop frictionless. What minimum distance, s, must the spring be compressed if the loop is to be completed? Express your answer in terms of parameters defined in the problem, and physical or mathematical constants. (*On Earth.*)



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Use the Energy Principle. Consider the Earth-spring-sled system. Elastic (spring) and gravitational potential energies change within the system. There are no internal forces changing the thermal energy. There are no external forces doing work on the system.

$$W_{\text{ext}} = \Delta K + \Delta U + \Delta E_{\text{th}} = (K_{\text{f}} - K_{\text{i}}) + (U_{s\text{f}} - U_{s\text{i}}) + (U_{g\text{f}} - U_{g\text{i}}) + 0$$

$$0 = \left(\tfrac{1}{2}mv_{\text{f}}^2 - \tfrac{1}{2}mv_{\text{i}}^2\right) + \left(\tfrac{1}{2}k\left(\Delta s_{\text{f}}\right)^2 - \tfrac{1}{2}k\left(\Delta s_{\text{i}}\right)^2\right) + \left(mgh_{\text{f}} - mgh_{\text{i}}\right)$$

Let the zero of height be at ground level (the initial position). The positive direction must be up and the zero of spring compression be at the relaxed (final) position to use the expressions above. The initial compression of the spring is s, and the initial speed of the car is zero. The height of the car at the top of the loop is $2R$.

$$0 = \left(\tfrac{1}{2}mv_{\text{f}}^2 - 0\right) + \left(0 - \tfrac{1}{2}ks^2\right) + \left(mg2R - 0\right)$$

The minimum speed of the car at the top of the loop can be found from Newton's Second Law. Both the weight and the normal force are downward at that point. The car is moving in a circle whose center is below the top. When the car has its minimum speed to successfully loop the loop, the normal force at the top is zero.

$$\sum F_c = w + n = ma_c \qquad \Rightarrow \qquad mg + 0 = m\,\frac{v_{\text{f}}^2}{R} \qquad \Rightarrow \qquad v_{\text{f}}^2 = gR$$

Substitute these into the energy expression and solve for s:

$$0 = \tfrac{1}{2}m\left(gR\right) - \tfrac{1}{2}ks^2 + mg2R \qquad \Rightarrow \qquad \tfrac{1}{2}ks^2 = \tfrac{1}{2}mgR + mg2R \qquad \Rightarrow \qquad s = \sqrt{5mgR/k}$$

II. (16 points) A huge cannon is assembled a planet that does not rotate. This planet has mass M and radius R. The cannon fires a projectile with mass m straight up at speed v_0. The projectile goes so high that the acceleration of gravity at its peak is different from that at its launch point. What maximum height above the surface does the projectile reach, in terms of parameters defined in the problem, and physical or mathematical constants?

Use the Energy Principle. Choose the system to consist of the planet and the projectile.

$$W_{\text{ext}} = \Delta K + \Delta U + \Delta E_{\text{th}}$$

Once the projectile is launched, there are no external forces on the system, and there are no internal forces changing the thermal energy. The potential energy that changes in the system is that of Universal Gravitation.

$$0 = \left(\tfrac{1}{2}mv_{\text{f}}^2 - \tfrac{1}{2}mv_{\text{i}}^2\right) + \left(\frac{-GMm}{r_{\text{f}}} - \frac{-GMm}{r_{\text{i}}}\right) + 0$$

The projectile starts at the planet surface $r_{\text{i}} = R$. The initial speed of the projectile was given as v_0. It's speed is zero when it reaches its maximum height.

$$0 = \left(0 - \tfrac{1}{2}mv_0^2\right) + \left(\frac{-GMm}{r_{\text{f}}} - \frac{-GMm}{R}\right)$$

Solve for r_{f}. Note that the mass of the projectile cancels.

$$\tfrac{1}{2}mv_0^2 = \frac{GMm}{R} - \frac{GMm}{r_{\text{f}}} \qquad \Rightarrow \qquad \frac{v_0^2}{2GM} = \frac{1}{R} - \frac{1}{r_{\text{f}}}$$

and

$$\frac{1}{r_{\text{f}}} = \frac{1}{R} - \frac{v_0^2}{2GM} \qquad \Rightarrow \qquad r_{\text{f}} = \left(\frac{1}{R} - \frac{v_0^2}{2GM}\right)^{-1}$$

But r_{f} is the distance the projectile reaches from the planet's center. The height, h, is one planet-radius less than that.

$$h = r_{\text{f}} - R = \left(\frac{1}{R} - \frac{v_0^2}{2GM}\right)^{-1} - R$$

1. (6 points) The cannon in the problem above will next be used to launch a new projectile to the same height. This new projectile has mass $m/3$. With what speed v' should this new projectile be launched?

   Since the both the initial kinetic energy and the gravitational potential energy change depend linearly on the object's mass, it does not affect the required launch speed.

$$v' = v_0$$

III. (16 points) An object of mass m is traveling in the positive direction along the x axis with speed v. It explodes into two pieces at the instant it reaches the origin, with negligible loss of mass. Afterward, one piece, with mass $m/3$, is found to be traveling in the positive direction along the y axis with speed $v/2$. What is the **velocity** of the other piece, with mass $2m/3$, in terms parameters defined in the problem, and physical or mathematical constants?

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There are no net external forces on a system consisting of the object, so linear momentum is conserved. (The explosion converts chemical potential energy into kinetic energy, so mechanical energy is **not** conserved.)

Let the piece of mass $m/3$ be piece "1", and the piece of mass $2m/3$ be piece "2".

Linear momentum is a vector quantity, and if is conserved, it will be conserved in each dimension. Looking first at the y direction

$$p_{yi} = p_{yf} \qquad \Rightarrow \qquad 0 = m_1 v_{1yf} + m_2 v_{2yf} = \frac{m}{3}\frac{v}{2} + \frac{2m}{3} v_{2yf} \qquad \Rightarrow \qquad v_{2yf} = -\frac{3}{2}\left(\frac{v}{6}\right) = -\frac{v}{4}$$

Looking next at the x direction

$$p_{xi} = p_{xf} \qquad \Rightarrow \qquad mv = m_1 v_{1xf} + m_2 v_{2xf} = 0 + \frac{2m}{3} v_{2xf} \qquad \Rightarrow \qquad v_{2xf} = \frac{3}{2} v$$

Now that the two components of the velocity are known

$$\vec{v}_{2f} = \frac{3}{2} v\, \hat{\imath} - \frac{1}{4} v\, \hat{\jmath}$$

Although it is not necessary, it is acceptable to find the velocity in polar form

$$\left|\vec{v}_{2f}\right| = \sqrt{v_{2xf}^2 + v_{2yf}^2} = \sqrt{\left(\frac{3}{2}v\right)^2 + \left(-\frac{1}{4}v\right)^2} = \frac{\sqrt{37}}{4} v = 1.52v$$

and

$$\tan\phi = \frac{v_{2yf}}{v_{2xf}} \qquad \Rightarrow \qquad \phi = \tan^{-1}\left(\frac{-v/4}{3v/2}\right) = \tan^{-1}(-1/6) = -9.5^\circ$$

So

$$\vec{v}_{2f} = 1.52v \;@\; -9.5^\circ$$

2. (6 points) In the problem above, how is the impulse from the explosion on the piece with mass $2m/3$ related to that on the piece with mass $m/3$?

The impulse on the piece with mass $2m/3$ is ...

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equal in magnitude and in the opposite direction to that on the piece with mass $m/3$.

Newton's Third Law assures us that the forces on the two objects are equal in magnitude and in the opposite direction. These forces must act for the same time.

3. (8 points) Two blocks, one with mass m and one with mass $2m$, are traveling along level frictionless tracks with the same momenta. Identical applied forces $\vec{F}_A$ will be used to bring each block to a stop. Compare the time and distance required to stop the blocks.

   Just as you do with "Choose one of each" questions in class, express your answer as single two-digit number, with your choices in numeric order. For example, if both the time and distance to stop the block with mass $2m$ is less than that to stop the block with mass m, your answer is "14".

   1 The time to stop the block with mass $2m$ is **less than** that to stop the block with mass m.

   2 The time to stop the block with mass $2m$ is **the same as** that to stop the block with mass m.

   3 The time to stop the block with mass $2m$ is **greater than** that to stop the block with mass m.

   4 The distance to stop the block with mass $2m$ is **less than** that to stop the block with mass m.

   5 The distance to stop the block with mass $2m$ is **the same as** that to stop the block with mass m.

   6 The distance to stop the block with mass $2m$ is **greater than** that to stop the block with mass m.

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   Since the two blocks have the same momentum, stopping them is the same momentum change. Therefore, each must have the same impulse $\vec{J} = \int \vec{F}\,dt = \vec{F}\,\Delta t = \Delta\vec{p}$ on it. Since the applied forces are the same, the same time is required for each.

   Since the two blocks have the same momentum, the block with mass $2m$ must have half the speed, and thus half the kinetic energy of the block with mass m. Therefore, only half the work $W_{\text{ext}} = \int \vec{F}\cdot d\vec{s} = F\,\Delta s = \Delta K$ must be done on the block with mass $2m$. Since the applied forces are the same, the block with mass $2m$ stops in less distance.

   These are items 2 and 4, making the answer

   24

4. (8 points) Two particles collide, one of which was moving, and the other was initially at rest. Consider these results of such a collision:

   i. *Both* particles are at rest.
   ii. *One* particle is at rest.
   iii. *Neither* particle is at rest.

   Which of those results are possible?

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   The particles are involved in a *collision*, so their interaction is brief, intense, and external forces may be neglected during the period of interaction. Thus, linear momentum is conserved.

   Since only one particle was moving before the collision, the total momentum of the two-particle system must be non-zero. It is impossible for both particles to be at rest after the collision, as the total momentum of the system would be zero in that case. Possible results of the collision, then, are

   Results *ii* and *iii* only.

5. (8 points) Bill and Susan are each standing on identical skateboards (with ideal ball bearings), initially at rest on level ground. Bill weighs three times as much as Susan. He pushes horizontally on Susan's back, causing her to start moving away from him. Immediately after Bill stops pushing, and in the reference frame of the Earth, ...

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The system consisting of Bill and Susan has no net external forces on it. Momentum, which was zero before Bill's push, must be conserved.

Susan and Bill are moving away from each other, and Susan's speed is three times that of Bill.

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6. (8 points) The potential energy, PE, of a system depends on the location x of an object within it, according to the graph. The total mechanical energy, TE, is constant. Rank the speed of the object at each of the labeled points i through v.

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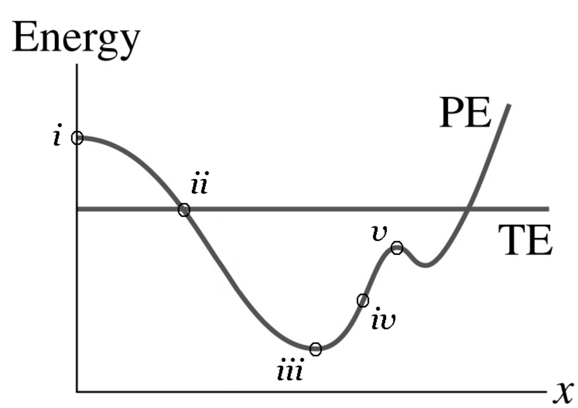
Since the total mechanical energy is the sum of the kinetic energy and the potential energy, the kinetic energy must be the difference between the total mechanical energy and the potential energy. The greatest difference between the total mechanical energy and the potential energy is at iii. The difference at ii is zero, the minimum, since the kinetic energy, $K = \frac{1}{2}mv^2$, cannot be negative. With a constant mass, the greatest speed corresponds to the greatest kinetic energy.

$iii > iv > v > ii$. Object cannot be at i.



7. (8 points) The potential energy of a system depends on the location of an object within it, according to the graph. What is the force acting on an object when it is at $x = 11\,\text{m}$?

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Force and potential energy are related by

$$F_s = -\frac{dU}{ds}$$

so the force is the opposite of the slope of the graph at $x = 11\,\text{m}$.

The graph is approximately linear from $x = 10\,\text{m}$ to $x = 12\,\text{m}$, and rises from $U = 10\,\text{J}$ to $U = 20\,\text{J}$. The force can be approximated by

$$F_s \approx -\frac{\Delta U}{\Delta s} = -\frac{20\,\text{J} - 10\,\text{J}}{12\,\text{m} - 10\,\text{m}} = -5\,\text{N}$$