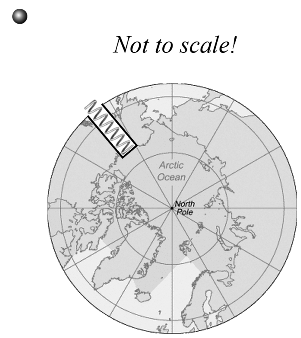

I. (16 points) A 444 kg asteroid is approaching Earth! It is detected when 19.1×10^3 km *above the Earth's surface*, traveling at 29.8 km/s. Let's protect the Earth with a giant spring that extends 1.25 km above the Earth's surface. If we're to bring the asteroid to a stop right at the surface, what spring constant will be needed? Note that Earth has mass 5.97×10^{24} kg and radius 6.38×10^3 km, and remember to neglect drag.



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Use the Energy Principle.

$$W_{\text{ext}} = \Delta K + \Delta U + \Delta E_{\text{th}}$$

Choose a system consisting of the Earth, the asteroid, and the spring. No external forces do work on that system ($W_{\text{ext}} = 0$). No internal forces increase the thermal energy of that system ($\Delta E_{\text{th}} = 0$). The kinetic energy change of the Earth is negligible. There are potential energy changes due to the internal spring force, and the internal gravitational force. Note that g is not constant over the distance traveled by the asteroid, so $U_g = mgh$ cannot be used. Let M represent the mass of the Earth, and m represent the mass of the asteroid.

$$0 = \left(\tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2\right) + \left(\frac{-GMm}{r_f} - \frac{-GMm}{r_i}\right) + \left(\tfrac{1}{2}k\,\Delta s_f^2 - \tfrac{1}{2}k\,\Delta s_i^2\right) + 0$$

The final speed of the asteroid is zero. The spring is not compressed before the asteroid hits it.

$$0 = \left(0 - \tfrac{1}{2}mv_i^2\right) + \left(\frac{-GMm}{r_f} - \frac{-GMm}{r_i}\right) + \left(\tfrac{1}{2}k\,\Delta s_f^2 - 0\right) \qquad \Rightarrow \qquad \tfrac{1}{2}mv_i^2 + GMm\left(\frac{1}{r_f} - \frac{1}{r_i}\right) = \tfrac{1}{2}k\,\Delta s_f^2$$

Note that, since the asteroid is detected 19.1×10^3 km *above the Earth's surface*, the initial distance from the Earth's center is

$$r_i = 19.1 \times 10^3\,\text{km} + 6.38 \times 10^3\,\text{km} = 25.48 \times 10^3\,\text{km}$$

Solve for the spring constant k, remembering to convert km to m.

$$\begin{aligned} k &= \left[v_i^2 + 2GM\left(\frac{1}{r_f} - \frac{1}{r_i}\right)\right]\left(\frac{m}{\Delta s_f^2}\right) \\ &= \Bigg[\left(29.8 \times 10^3\,\text{m/s}\right)^2 \\ &\quad + 2\left(6.673 \times 10^{-11}\,\frac{\text{N}\cdot\text{m}^2}{\text{kg}^2}\right)\left(5.97 \times 10^{24}\,\text{kg}\right)\left(\frac{1}{6.38 \times 10^6\,\text{m}} - \frac{1}{25.48 \times 10^6\,\text{m}}\right)\Bigg]\left(\frac{444\,\text{kg}}{\left(1.25 \times 10^3\,\text{m}\right)^2}\right) \\ &= 2.79 \times 10^5\,\text{N/m} \end{aligned}$$

II. (16 points) A sled with mass m is sliding at speed v_0 along the frictionless level ground of Planet X. It encounters a rough patch of length d having a coefficient of kinetic friction μ_k with the sled. Then it descends frictionless hill of height h before encountering a second rough patch, identical to the first. If it emerges from this second rough patch with a speed $2v_0$, what is the gravitational acceleration g_x on Planet X? Express your answer in terms of parameters defined in the problem, and physical or mathematical constants.

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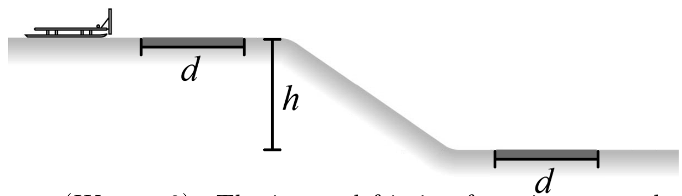

Use the Energy Principle.

$$W_{\text{ext}} = \Delta K + \Delta U + \Delta E_{\text{th}}$$

Choose a system consisting of the Earth and the sled. No external forces do work on that system ($W_{\text{ext}} = 0$). The internal friction force increases the thermal energy of that system. The kinetic energy change of the Earth is negligible. There is a potential energy change due to the internal gravitational force.

$$0 = \left(\tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2\right) + (mg_x h_f - mg_x h_i) + f_k\,(2d)$$

The kinetic friction force is $f_k = \mu_k n$, and a quick Free Body Diagram with Newton's Second Law shows us that $n = mg_x$. If we let the final height of the sled be zero, then its initial height is h.

$$0 = \left(\tfrac{1}{2}m\,(2v_0)^2 - \tfrac{1}{2}mv_0^2\right) + (0 - mg_x h) + 2\mu_k m g_x d$$

Solve for g_x. Note that the mass of the sled cancels.

$$0 = \tfrac{1}{2}\left(4v_0^2 - v_0^2\right) - g_x h + 2\mu_k g_x d \qquad \Rightarrow \qquad g_x h - 2\mu_k g_x d = 3v_0^2/2 \qquad \Rightarrow \qquad g_x = \frac{3v_0^2/2}{h - 2\mu_k d}$$

1. (6 points) What if, in the problem above, there was no rough patch before the sled reached the hill, but instead a rough patch of length $2d$ below the hill, having the same coefficient of kinetic friction μ_k? The speed of the sled emerging from the rough patch would be ...

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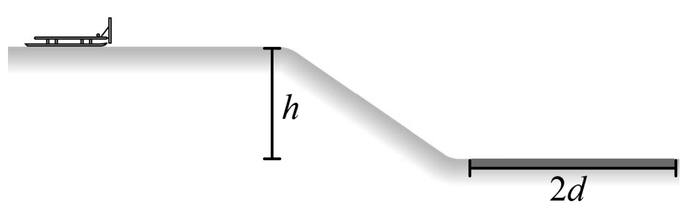

The energy transformations in this system are identical to those in the problem above. The sled emerges with speed

$2v_0$

2. (6 points) An object of mass M is traveling in the $+x$ direction when it explodes into two pieces with negligible loss of mass. What is conserved in this process?

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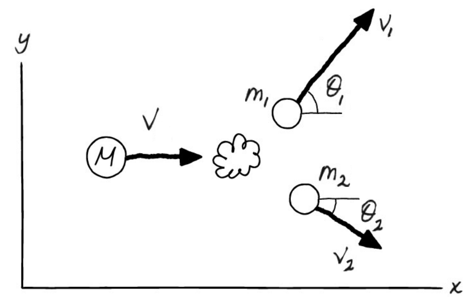
The explosion converts chemical potential energy to kinetic energy, so Kinetic Energy is not conserved. But as the forces of the explosion are internal to the object, what's conserved is

Momentum.



III. (16 points) In the problem above, the object of mass $M = 12\,\text{kg}$ is traveling in the $+x$ direction at $V = 9.0\,\text{m/s}$ before the explosion. Afterward, a piece with mass $m_1 = 8.0\,\text{kg}$ travels at speed $v_1 = 14\,\text{m/s}$ in a direction $\theta_1 = 52°$ from the original direction of travel, as shown. What is the **velocity** of the piece with mass $m_2 = 4.0\,\text{kg}$?

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Momentum is conserved on each axis.

$$P_{ix} = P_{fx} \qquad \Rightarrow \qquad MV = m_1 v_{1fx} + m_2 v_{2fx}$$

So

$$v_{2fx} = \frac{MV - m_1 v_{1f}\cos\theta_1}{m_2} = \frac{(12\,\text{kg})\,(9.0\,\text{m/s}) - (8.0\,\text{kg})\,(14\,\text{m/s})\cos 52°}{4.0\,\text{kg}} = 9.76\,\text{m/s}$$

And

$$P_{iy} = P_{fy} \qquad \Rightarrow \qquad 0 = m_1 v_{1yx} + m_2 v_{2yx}$$

So

$$v_{2fx} = \frac{-m_1 v_{1f}\sin\theta_1}{m_2} = \frac{-\,(8.0\,\text{kg})\,(14\,\text{m/s})\sin 52°}{4.0\,\text{kg}} = -22.1\,\text{m/s}$$

Then

$$\vec{v}_{2f} = 9.8\,\hat{\imath} - 22\,\hat{\jmath}\,\text{m/s}$$

Or

$$v_{2f} = \sqrt{v_{2fx}^2 + v_{2fy}^2} = \sqrt{(9.76\,\text{m/s})^2 + (-22.1\,\text{m/s})^2} = 24\,\text{m/s}$$

and

$$\phi = \tan^{-1}\frac{v_{2fy}}{v_{2fx}} = \tan^{-1}\frac{9.76\,\text{m/s}}{-22.1\,\text{m/s}} = -66°$$

So

$$\vec{v}_{2f} = 24\,\text{m/s} \quad @ \quad -66°$$

3. (7 points) A block of mass M is at rest on a level frictionless table. It is struck by a horizontally traveling bullet of mass m. The bullet **does not** embed itself in the block! Instead, it passes through to emerge from the other side. Consider a system consisting of the bullet and the block. How does the horizontal component of its momentum change in this case? (*On Earth.*)

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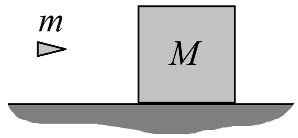

The interaction forces between the bullet and the block are internal to the bullet/block system. Momentum is conserved, meaning

The change is zero.

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4. (7 points) A system has a potential energy, U, as a function of position x of a particle within it, as shown. Where, in the range 0 to 8 m, does the particle experience maximum force magnitude?

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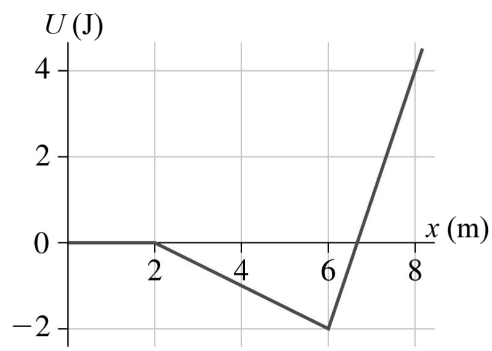

Force and potential energy are related by

$$F_x = -\frac{\delta U}{\delta x}$$

so the force on the particle is the opposite of the slope of the graph. Maximum force magnitude is at maximum slope,

Between 6 and 8 m.

5. (7 points) A 2.0 kg object is traveling in the $+x$ direction at 3.0 m/s. At time $t = 0$, it becomes subject to the force shown. What is the object's velocity at time $t = 12\,\text{s}$?

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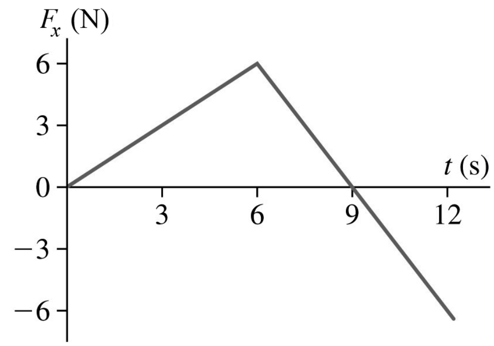


The Impulse-Momentum Theorem is

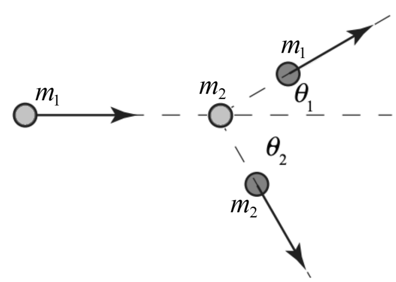
$$\vec{J} = \int \vec{F}\,dt = \Delta\vec{p} = m\,\Delta\vec{v}$$

so the change in the object's momentum is the area under the force-time curve. Letting signs indicate direction, there are $+27\,\text{N·s}$ of impulse between zero and 9 s, and $-9\,\text{N·s}$ between 9 s and 12 s, for a total of $+18\,\text{N·s}$. So

$$J = mv_f - mv_i \qquad \Rightarrow \qquad v_f = \frac{J + mv_i}{m}$$

$$= \frac{+18\,\text{N·s} + (2.0\,\text{kg})\,(+3.0\,\text{m/s})}{2.0\,\text{kg}} = +12\,\text{m/s}$$

6. (7 points) An object of mass m_1 is moving with velocity $\vec{v}_{1\text{i}}$ and momentum $\vec{p}_{1\text{i}}$. It strikes a stationary object of mass m_2 with a glancing blow in an elastic collision. Afterward, object m_1 moves with velocity $v_{1\text{f}}$ at an angle θ_1 from its original direction. Object m_2 moves with velocity $v_{2\text{f}}$ at an angle θ_2 from object m_1's original direction. If m_1 is **not equal** to m_2, which sketch, if any, could represent a valid statement?



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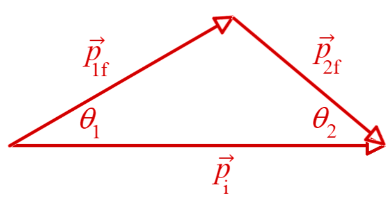
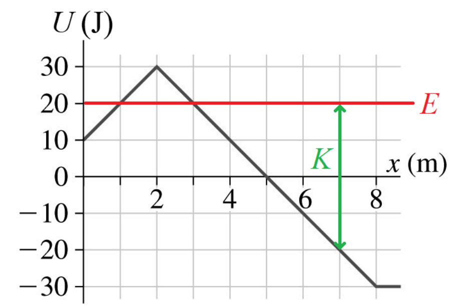
This is a collision, so momentum is conserved,

$$\vec{p}_\text{i} = \vec{p}_{1\text{f}} + \vec{p}_{2\text{f}}$$

The graphical representation of this relationship is



7. (6 points) A system has a potential energy, U, as a function of position x of a 2.0 kg particle within it, as shown. At one instant, the particle is observed to be traveling in the negative direction through $x = +7$ m at speed $2\sqrt{10}$ m/s. If the particle has a turning point, where is it?



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When the particle is at $x = +7$ m it has a kinetic energy

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}\,(2.0\,\mathrm{kg})\left(2\sqrt{10}\,\mathrm{m/s}\right)^2 = 40\,\mathrm{J}$$

Marking that on the graph shows that the total energy of the system is 20 J. Where the total energy E and potential energy U are the same, the kinetic energy must be zero. Where the kinetic energy is zero, the particle's speed must be zero. It stops and turns around

At $x = +3$ m.

8. (6 points) In the problem above, if the particle reaches position $x = +5$ m, what is its speed there?

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When the particle is at $x = +7$ m it has a kinetic energy

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}\,(2.0\,\mathrm{kg})\left(2\sqrt{10}\,\mathrm{m/s}\right)^2 = 40\,\mathrm{J}$$

Marking that on the graph shows that the total energy of the system is 20 J. When the particle is at $x = +5$ m the system's potential energy is zero, so the particle's kinetic energy must be 20 J. Its speed at that point is

$$K = \tfrac{1}{2}mv^2 \qquad \Rightarrow \qquad v = \sqrt{2K/m} = \sqrt{\frac{2\,(20\,\mathrm{J})}{2.0\,\mathrm{kg}}} = 2\sqrt{5}\,\mathrm{m/s}$$