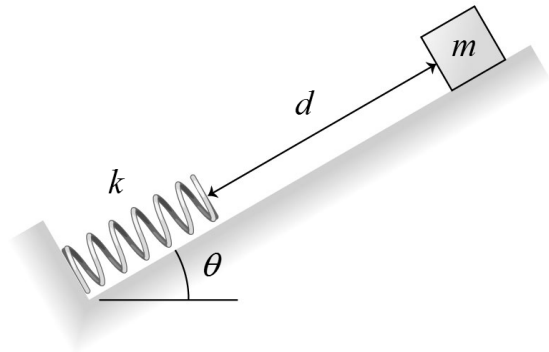


I. (16 points) A block of mass $m = 0.010$ kg is released from rest on a frictionless surface that makes an angle $\theta = 30.0^\circ$ with respect to the horizontal. From what distance d should block be released in order to compress the nonlinear spring a maximum distance $x = 2.0$ cm? The force exerted by the nonlinear spring depends on compression distance according to

$$F(x) = kx^3$$

where $k = 3.8 \times 10^5$ N/m³. (On Earth.)



Use the Energy Principle,

$$W_{\text{ext}} = \Delta K + \Delta U + \Delta E_{\text{th}}$$

Since the spring is nonlinear, the potential energy expression for a linear spring, $U_s = \frac{1}{2}k(\Delta s)^2$, isn't appropriate. So I'll choose a system consisting of the block and the Earth, making the spring's force external. The gravitational force is internal and conservative, changing the system's potential energy. Although there is kinetic energy in the system at times, there is none at the instant the block is released, nor is there any at the instant the spring reaches maximum compression ($\Delta K = 0$). There are no internal forces changing the thermal energy of the system ($\Delta E_{\text{th}} = 0$).

$$\int \vec{F} \cdot d\vec{s} = 0 + \Delta U + 0 \quad \Rightarrow \quad \int F \cos \phi ds = mgh_f - mgh_i$$

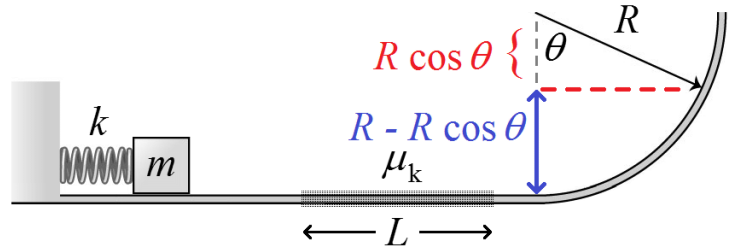
As the block compresses the spring, it moves down the plane while the spring's force is directed up the plane. Therefore, $\phi = 180^\circ$ and $\cos \phi = -1$. I'll choose the zero of height at the maximum compression of the spring, making $h_f = 0$, and $h_i = (d + x) \sin \theta$.

$$-\int_0^x kx^3 dx = 0 - mg(d + x) \sin \theta \quad \Rightarrow \quad \left. \frac{1}{4}kx^4 \right|_0^x = \frac{1}{4}kx^4 - \frac{1}{4}k0^4 = \frac{1}{4}kx^4 = mg(d + x) \sin \theta$$

Solve for d .

$$\begin{aligned} \frac{kx^4}{4mg \sin \theta} = d + x & \quad \Rightarrow \quad d = \frac{kx^4}{4mg \sin \theta} - x \\ & = \frac{(3.8 \times 10^5 \text{ N/m}^3) (0.020 \text{ m})^4}{4 (0.010 \text{ kg}) (9.81 \text{ m/s}^2) \sin 30.0^\circ} - 0.020 \text{ m} = \mathbf{0.29 \text{ m}} \end{aligned}$$

- II. (16 points) Using a spring with Hooke's Law Constant $k = 12 \text{ N/m}$ that is compressed a distance $x = 0.20 \text{ m}$, a mass $m = 0.10 \text{ kg}$ is launched along a track that is frictionless except for a rough section of length $L = 0.20 \text{ m}$ and coefficient of kinetic friction $\mu_k = 0.20$. After scraping across the rough section, the mass slides up a circular section of radius $R = 2.0 \text{ m}$, reaching a maximum angle θ with respect to the vertical before sliding back down again. Find the angle θ . You may assume that the mass does make it across the rough section. (On Earth.)



Use the Energy Principle,

$$W_{\text{ext}} = \Delta K + \Delta U + \Delta E_{\text{th}}$$

I'll choose a system consisting of the block, the spring, the Earth, and the track. With that choice, there are no external forces, so external forces do no work ($W_{\text{ext}} = 0$). Both the spring force and the gravitational force are conservative internal forces that change the system's potential energy. The track's rough patch transforms mechanical energy to thermal energy. Although there is kinetic energy in the system at times, there is none at the instant the block is launched, nor is there any at the instant the block reaches its maximum angle ($\Delta K = 0$).

$$0 = 0 + (mgh_f - mgh_i) + \left(\frac{1}{2}ks_f^2 - \frac{1}{2}ks_i^2\right) + f_k d$$

I'll choose $h = 0$ at the initial height of the block ($h_i = 0$). The spring is not compressed at the instant the block reaches its maximum angle ($s_f = 0$). However, it is compressed a distance $s_i = x$ initially. The block traverses a distance $d = L$ through the rough patch. The force of kinetic friction is $f_k = \mu_k n$. So

$$0 = (mgh_f - 0) + \left(0 - \frac{1}{2}kx^2\right) + \mu_k n L$$

The block does not accelerate vertically while it is traversing the rough patch, so Newton's Second Law tells us $n = mg$. The height of the block at maximum angle θ is related to R as shown above, so

$$0 = mg(R - R \cos \theta) - \frac{1}{2}kx^2 + \mu_k mgL$$

Solve for θ .

$$mgR \cos \theta = mgR - \frac{1}{2}kx^2 + \mu_k mgL \quad \Rightarrow \quad \cos \theta = \frac{mgR + \mu_k mgL - \frac{1}{2}kx^2}{mgR} = 1 + \frac{\mu_k mgL - \frac{1}{2}kx^2}{mgR}$$

$$\theta = \cos^{-1} \left[1 + \frac{(0.20 \text{ m})(0.10 \text{ kg})(9.81 \text{ m/s}^2)(0.20 \text{ m}) - \frac{1}{2}(12 \text{ N/m})(0.20 \text{ m})^2}{(0.10 \text{ kg})(9.81 \text{ m/s}^2)(2.0 \text{ m})} \right] = 26^\circ$$

1. (6 points) In the problem above, will the block eventually stop and remain motionless? If so, where?

Each time the block traverses the rough section, kinetic energy is transformed to thermal energy. In other regions, kinetic energy may be transformed to elastic or gravitational potential energy, but that transformation is reversible. Even though the block stops momentarily at its maximum height, or when the spring is at maximum compression, it will not remain motionless.

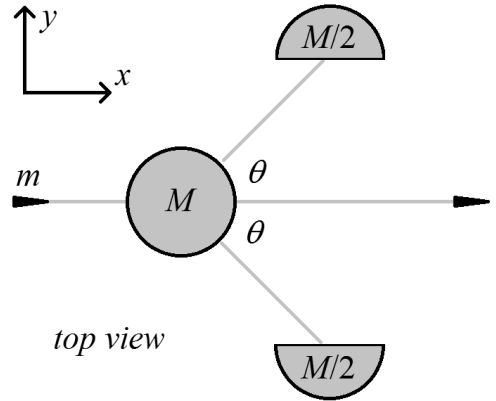
Eventually, all the block's kinetic energy will be transformed to thermal energy. This must happen while it is in the rough section, so

Yes, it will stop and remain motionless somewhere in the rough section.

2. (6 points) A wooden cylinder of mass M is at rest on a horizontal frictionless surface, as shown from above. A bullet of mass m is traveling to the right in the $+x$ direction with initial speed v_i . It passes through the cylinder, emerging with final speed $v_f < v_i$. The cylinder splits in half, with each piece travelling in a direction θ from the bullet's direction. Considering a system consisting of the bullet and the cylinder, what must be conserved in this process?

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 Kinetic energy will be transformed to thermal (and probably acoustic) energy as the bullet splits the wooden cylinder. Kinetic energy will not be conserved. However, there are no net external forces on the bullet-block system in the x or y directions. The conserved quantities are

Both the x and y components of momentum.



- III. (16 points) In the problem above, the wooden cylinder has mass $M = 1.40$ kg and the bullet has mass $m = 0.028$ kg. The initial speed of the bullet is $v_i = 380$ m/s and its final speed is $v_f = 220$ m/s. If the angle θ is 42° , what is the resulting speed of one half of the cylinder?

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 Consider the x component of momentum.

$$p_{xf} = p_{xi} \quad \Rightarrow \quad \frac{M}{2}V_{fx} + \frac{M}{2}V_{fx} + mv_{fx} = mv_{ix}$$

where V_f is the speed of one half of the cylinder after it is split by the bullet. Note that the velocity of the bullet is in the x direction both before and after the collision. So

$$2 \left(\frac{M}{2} \right) V_f \cos \theta + mv_f = mv_i$$

Solve for V_f .

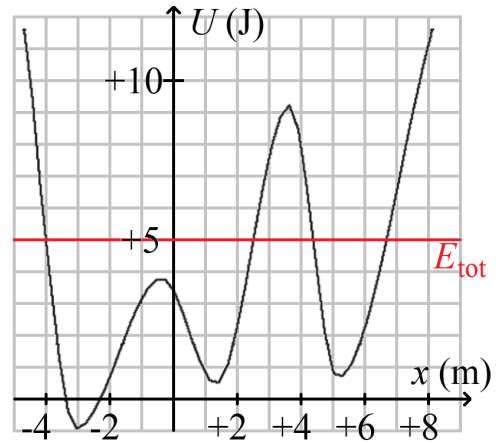
$$V_f = \frac{mv_i - mv_f}{M \cos \theta} = \frac{0.028 \text{ kg} (380 \text{ m/s} - 220 \text{ m/s})}{(1.40 \text{ kg}) \cos 42^\circ} = 4.3 \text{ m/s}$$

3. (8 points) A system has a potential energy U that depends on the position x of a particle within it, as shown. If the particle has a turning point at $x = -4.0$ m, does it have another turning point? If so, where?

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At a turning point, the particle has no kinetic energy, so the total energy E_{tot} and the potential energy U of the system are the same. With a turning point at $x = -4.0$ m, the total energy of the system must be +5 J, as marked on the graph. The other point at which the particle has zero kinetic energy, making the total and potential energies the same, is at +2.5 m. There are two other points at which the total and potential energies of the system are the same, but if the particle has a turning point at $x = -4.0$ m, it cannot reach those other points. So

Yes, its other turning point is at about +2.5 m.



4. (8 points) The force exerted by a frog muscle depends on velocity v of the contracting muscle according to

$$F(v) = F_0 (v_0^2 - v^2)$$

where $v_0 = 2.0$ cm/s, and $F_0 = 0.020$ N·s²/cm². At what contraction speed does this muscle deliver maximum power?

.....

Power is the time rate of energy transformation, $P = dE/dt$, but is conveniently $P = \vec{F} \cdot \vec{v}$. A contracting muscle will pull, so its force is in the direction its velocity, making $P = Fv$. At maximum power

$$\frac{dP}{dv} = 0 = \frac{d}{dv} [Fv] = \frac{d}{dv} [F_0 (v_0^2 - v^2) v] = \frac{d}{dv} [F_0 (v_0^2 v - v^3)] = F_0 (v_0^2 - 3v^2)$$

Solve for v .

$$F_0 (v_0^2 - 3v^2) = 0 \quad \Rightarrow \quad v_0^2 - 3v^2 = 0 \quad \Rightarrow \quad 3v^2 = v_0^2 \quad \Rightarrow \quad v = v_0/\sqrt{3}$$

So maximum power is delivered at

$$v = (2.0 \text{ cm/s})/\sqrt{3} = 1.2 \text{ cm/s}$$

(Note that this is not minimum power, which, by inspection, occurs when $v = v_0$.)

5. (8 points) A satellite is in a circular orbit at an altitude $3R_E$ above the Earth's surface, where R_E is the Earth's radius. Its thrusters fire, putting it in a new circular orbit at an altitude $2R_E$ above the Earth's surface (you may assume that the mass of fuel consumed in this process is negligible). Describe the change in the satellite's kinetic energy, and the change in the gravitational potential energy of the Earth-satellite system.

Note: g is NOT the same at an altitude of $3R_E$ and an altitude of $2R_E$. Remember that a positive value becoming less positive, and a negative value becoming more negative, are both "decreases", etc.

Since g is not the same at the two altitudes, the potential energy of the Earth-satellite system is

$$U_G = -G\frac{Mm}{r}$$

As r changes from $3R_E$ to $2R_E$, U_G becomes more negative, which is a decrease.

The kinetic energy can be found by applying Newton's Second Law to the satellite:

$$\sum F_c = F_G = ma_c \Rightarrow G\frac{Mm}{r^2} = m\frac{v^2}{r} \Rightarrow G\frac{Mm}{r} = mv^2 \Rightarrow G\frac{Mm}{2r} = \frac{1}{2}mv^2 = K$$

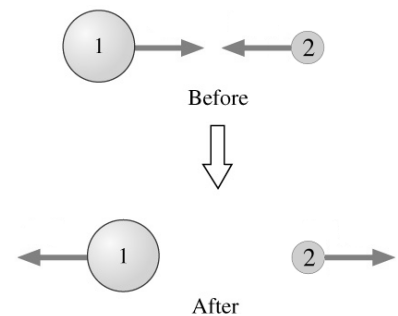
As r changes from $3R_E$ to $2R_E$, K becomes more positive, which is an increase.

Potential energy decreases. Kinetic energy increases.

6. (8 points) Object 1, with mass m_1 , is travelling to the right with 60 J of kinetic energy. Object 2, with mass m_2 , is travelling to the left with 40 J of kinetic energy. They undergo a perfectly elastic collision, after which object 1 is travelling to the left with 30 J of kinetic energy and object 2 is travelling to the right. If $m_1 > m_2$, what is the kinetic energy of object 2 after the collision?

Since the collision is elastic, kinetic energy is conserved. Before the collision, there is a total of $60 \text{ J} + 40 \text{ J} = 100 \text{ J}$ in the two-object system. Afterward, object 1 has 30 J, so object 2 must have 70 J. Kinetic energy is a non-negative scalar, so the directions are irrelevant.

+70 J



7. (8 points) A 2.0 kg object is travelling with a *speed* of 40 m/s in the $-x$ direction, when it becomes subject to a force that depends on time as shown. What is the *velocity* of the object after 10 s?

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Impulse changes the momentum of an object.

$$\vec{J} = \Delta p \quad \Rightarrow \quad \int \vec{F} dt = mv_f - mv_i$$

Impulse is the area under a graph of force as a function of time. In this case, the negative area from 0 s to 2.5 s cancels the positive area from 2.5 s to 5 s. The total area under the curve is the area from 5 s to 10 s, or +25 N·s.

The initial momentum of the object is $\vec{p} = m\vec{v} = (2.0 \text{ kg})(-40 \text{ m/s}) = -80 \text{ N}\cdot\text{s}$. The change (impulse) is +25 N·s. So the final momentum is $-55 \text{ N}\cdot\text{s}$. Dividing by the mass of 2.0 kg yields a velocity of

$$-27.5 \text{ m/s}$$

