
I. (16 points) A block of mass $m = 3.3\,\mathrm{kg}$ is sliding at speed $v_0 = 6.6\,\mathrm{m/s}$ on a frictionless level surface. It then enters a region in which the coefficient of kinetic friction between the block and the surface is $\mu_k = 0.11$. After sliding a distance $d = 0.55\,\mathrm{m}$ in this region, it encounters a spring with Hooke's Law constant $k = 88\,\mathrm{N/m}$. What is the maximum compression of the spring? (*On Earth.* Note that friction between the block and the surface continues after the block hits the spring!)

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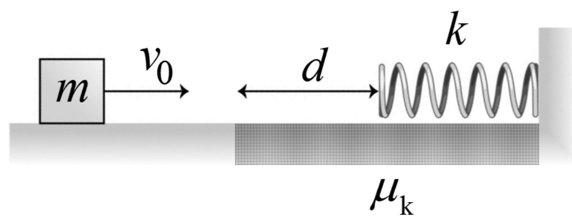


Use the Energy Principle,

$$W_{\mathrm{ext}} = \Delta K + \Delta U + \Delta E_{\mathrm{th}}$$

Choose a system consisting of the block, the spring, and the surface. External forces do no work on this system. The potential energy of the spring changes, and kinetic friction transforms mechanical energy to thermal energy, so

$$0 = \left(\tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2\right) + \left(\tfrac{1}{2}ks_f^2 - \tfrac{1}{2}s_i^2\right) + f_k\,\Delta x$$

The block is stationary ($v_f = 0$) at maximum compression. The spring is not compressed before the block reaches it ($s_i = 0$). The friction force, $f_k = \mu_k n = \mu_k mg$, acts on the block for a distance $\Delta x = d + s_f$. So

$$0 = \left(0 - \tfrac{1}{2}mv_i^2\right) + \left(\tfrac{1}{2}ks_f^2 - 0\right) + \mu_k mg\,(d + s_f)$$

This is quadratic in s_f:

$$\tfrac{1}{2}ks_f^2 + \mu_k mgs_f + \left(\mu_k mgd - \tfrac{1}{2}mv_i^2\right) = 0$$

so

$$s_f = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \qquad \text{where} \qquad A = \tfrac{1}{2}k = \tfrac{1}{2}\,(88\,\mathrm{N/m}) = 44\,\mathrm{N/m}$$

$$B = \mu_k mg = 0.11\,(3.3\,\mathrm{kg})\left(9.81\,\mathrm{m/s^2}\right) = 3.56\,\mathrm{N}$$

$$\begin{aligned} C &= \mu_k mgd - \tfrac{1}{2}mv_i^2 \\ &= 0.11\,(3.3\,\mathrm{kg})\left(9.81\,\mathrm{m/s^2}\right)(0.55\,\mathrm{m}) - \tfrac{1}{2}\,(3.3\,\mathrm{kg})\,(6.6\,\mathrm{m/s})^2 \\ &= 69.9\,\mathrm{N{\cdot}m} \end{aligned}$$

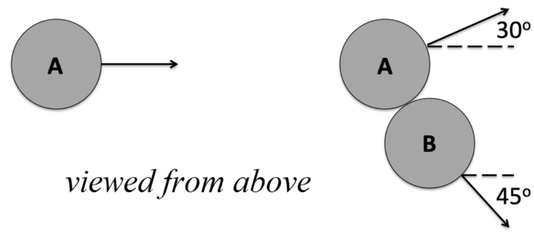
making

$$s_f = \frac{-3.56\,\mathrm{N} \pm \sqrt{(3.56\,\mathrm{N})^2 - 4\,(44\,\mathrm{N/m})\,(69.9\,\mathrm{N{\cdot}m})}}{2\,(44\,\mathrm{N/m})} = 1.2\,\mathrm{m} \quad \text{or} \quad -1.3\,\mathrm{m}$$

But when we set $\Delta x = d + s_f$, we implicitly defined s_f to be positive. The maximum compression of the spring, therefore, is

$$1.2\,\mathrm{m}$$

1. (6 points)An air hockey puck (A) of mass m is moving along the x-axis with speed v_0 when it hits an identical stationary puck (B). Puck A changes direction to move 30° above the x-axis and puck (B) moves 45° below the x-axis. Just with the information provided, and without any calculations, which statement do you know must be *incorrect*?



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In a perfectly inelastic collision, maximum kinetic energy is lost, and the two objects stick together. Pucks A and B don't stick together, so this statement must be *incorrect*:

The collision is perfectly inelastic.

II. (16 points) In the problem above, find the final speed of puck A. Express your answer in terms of m and v_0.

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There are no net external forces on the two-puck system, so momentum in it is conserved.

$$\vec{p}_i = \vec{p}_f \quad \Rightarrow \quad p_{ix} = p_{fx} \quad \text{and} \quad p_{iy} = p_{fy}$$

To the right, in the x direction:

$$p_{ix} = p_{fx} \quad \Rightarrow \quad m_A v_{Aix} + m_B v_{Bix} = m_A v_{Afx} + m_B v_{Bfx} \quad \Rightarrow \quad m v_0 + 0 = m v_{Af} \cos\theta_A + m v_{Bf} \cos\theta_B$$

Up the page, in the y direction:

$$p_{iy} = p_{fy} \quad \Rightarrow \quad m_A v_{Aiy} + m_B v_{Biy} = m_A v_{Afy} + m_B v_{Bfy} \quad \Rightarrow \quad 0 + 0 = m v_{Af} \sin\theta_A - m v_{Bf} \sin\theta_B$$

So

$$v_{Bf} = v_{Af} \frac{\sin\theta_A}{\sin\theta_B}$$

Substitute this into the x expression,

$$m v_0 = m v_{Af} \cos\theta_A + m \left(v_{Af} \frac{\sin\theta_A}{\sin\theta_B} \right) \cos\theta_B$$

and solve for v_{Af}. Note that, since $\theta_B = 45°$, $\sin\theta_B = \cos\theta_B$.

$$v_0 = v_{Af} \left[\cos\theta_A + \left(\frac{\sin\theta_A}{\sin\theta_B} \right) \cos\theta_B \right] = v_{Af} \left[\cos\theta_A + \sin\theta_A \right]$$

Then

$$v_{Af} = \frac{v_0}{\cos\theta_A + \sin\theta_A} = \frac{v_0}{\cos 30° + \sin 30°} = \frac{v_0}{\sqrt{3}/2 + 1/2} = \frac{2v_0}{\sqrt{3}+1} = 0.73 v_0$$

III. (16 points) An object is resting on a frictionless surface and is subject to the following force:

$$\vec{F} = Ay\cos\left(\frac{\pi x}{B}\right)\hat{\imath} + Ax\sin\left(\frac{\pi y}{B}\right)\hat{\jmath}$$

where $A = 1.0\,\mathrm{N/m}$ and $B = 1.0\,\mathrm{m}$. The object may move from the origin to $(1.0\,\mathrm{m},\ 1.0\,\mathrm{m})$ by one of two paths:

Path a: Origin to $(1.0\,\mathrm{m},\ 0.0\,\mathrm{m})$, then $(1.0\,\mathrm{m},\ 0.0\,\mathrm{m})$ to $(1.0\,\mathrm{m},\ 1.0\,\mathrm{m})$
Path b: Origin to $(0.0\,\mathrm{m},\ 1.0\,\mathrm{m})$, then $(0.0\,\mathrm{m},\ 1.0\,\mathrm{m})$ to $(1.0\,\mathrm{m},\ 1.0\,\mathrm{m})$

If the object follows path b, the force does zero work on the object. Calculate the work done by the force on the object if it follows path a.

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Work is related to force and displacement.

$$W = \int \vec{F}\cdot\vec{ds} = \int F_x\,dx + \int F_y\,dy$$

For the first part of the path, origin to $(1.0\,\mathrm{m},\ 0.0\,\mathrm{m})$, there is no component of displacement in the y direction, so (remembering that y is the constant $0.0\,\mathrm{m}$)

$$W_1 = \int F_x\,dx = \int_{0\,\mathrm{m}}^{1\,\mathrm{m}} Ay\cos\left(\frac{\pi x}{B}\right)\,dx = \int_{0\,\mathrm{m}}^{1\,\mathrm{m}} A\,(0.0\,\mathrm{m})\cos\left(\frac{\pi x}{B}\right)\,dx = 0\,\mathrm{J}$$

For the second part of the path, $(1.0\,\mathrm{m},\ 0.0\,\mathrm{m})$ to $(1.0\,\mathrm{m},\ 1.0\,\mathrm{m})$, there is no component of displacement in the x direction, so (remembering that x is the constant $1.0\,\mathrm{m}$)

$$\begin{aligned}
W_2 &= \int F_y\,dy = \int_{0\,\mathrm{m}}^{1\,\mathrm{m}} Ax\sin\left(\frac{\pi y}{B}\right)\,dy = Ax\,\frac{B}{\pi}\int_{0\,\mathrm{m}}^{1\,\mathrm{m}} \sin\left(\frac{\pi y}{B}\right)\frac{\pi}{B}\,dy \\
&= Ax\,\frac{B}{\pi}\left[-\cos\left(\frac{\pi y}{B}\right)\right]_{0\,\mathrm{m}}^{1\,\mathrm{m}} = -A\,(1.0\,\mathrm{m})\,\frac{B}{\pi}\left[\cos\left(\frac{\pi\,(1.0\,\mathrm{m})}{B}\right) - \cos\left(\frac{\pi\,(0.0\,\mathrm{m})}{B}\right)\right] \\
&= -\,(1.0\,\mathrm{N/m})\,(1.0\,\mathrm{m})\,\frac{(1.0\,\mathrm{m})}{\pi}\left[\cos\left(\frac{\pi\,(1.0\,\mathrm{m})}{1.0\,\mathrm{m}}\right) - \cos\left(\frac{\pi\,(0.0\,\mathrm{m})}{1.0\,\mathrm{m}}\right)\right] \\
&= \frac{-\,(1.0\,\mathrm{N\cdot m})}{\pi}\Big[\cos(\pi) - \cos(0)\Big] = \frac{-\,(1.0\,\mathrm{N\cdot m})}{\pi}\Big[-1-1\Big] = \frac{2.0}{\pi}\,\mathrm{J} = 0.64\,\mathrm{J}
\end{aligned}$$

Since no work was done on the first part of the path, the total work done by the force on the object following path a is

$$0.64\,\mathrm{J}$$

2. (6 points) The work along path a above is different than that of path b. What is true about this force?

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The work done by a conservative force on an object as it moves between two points is independent of path, so

This force is not conservative, because there are two paths along which the work differs.

3. (8 points) A truck of mass M is moving with speed V. A tennis ball, mass m, traveling in the opposite direction with speed v, hits the front of the truck head-on. What will be the velocity of the ball after the collision? Assume that the mass of the truck is much larger than that of the tennis ball ($M \gg m$).

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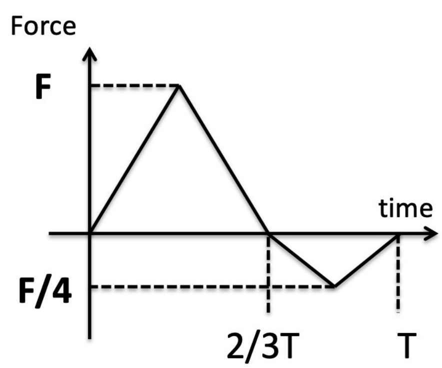
The ball must be moving in the same direction as the truck. Otherwise, it would have to pass through the truck, which is unphysical.

In the frame of the truck the ball is approaching at speed $V + v$ before the collision. We expect the ball to "bounce", so, again *in the frame of the truck*, after the collision the ball will be receding from the truck at speed $V + v$.

But in the frame of us, who watched this event and measured the speeds, the truck has an unchanging speed V. If the ball is going $V + v$ faster than that, then

The ball will be moving in the same direction of the truck with speed $2V + v$

4. (8 points) An object is subject to the time-varying force shown. Calculate the average force on the object over a time T.



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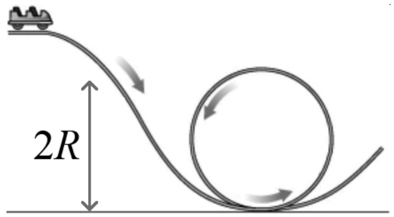
The average force is the constant force that would provide the same impulse as the given force. As $\vec{J} = \int \vec{F}\, dt$, the average force must provide the same area under the graph of force as a function of time that the given force does. This graph consists of two triangles. Letting signs indicate directions, the impulse is

$$J = \tfrac{1}{2}\left(\frac{2T}{3}\right)(F) + \tfrac{1}{2}\left(\frac{T}{3}\right)(-F/4) = FT/3 - FT/24 = 7FT/24$$

A rectangle with area $7FT/24$ and width T would have height

$7F/24$

5. (8 points) The roller-coaster in the illustration is truly a "coaster" — the car speeds up as it descends on its frictionless track, and slows down again as it ascends. If the care is released from rest, from what height h **must** the it start to successfully complete the loop? *(On Earth.)*



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Consider conservation of kinetic energy and gravitational potential energy in a system consisting of the car and the Earth.

If the car started lower than the top of the loop, then it wouldn't get to the top of the loop.

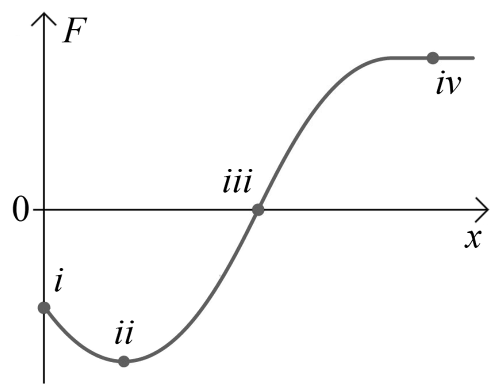
If the car started even with the top of the loop, then it would stop at the top, and fall straight down. That's bad.

If the car started slightly above the top of the loop, it would reach the top, but the gravitational force on it would be greater than that required to move the car in a circle. The car would leave the track and move in a parabola (projectile motion). That's bad, too.

But if the car started high enough, it would reach the top of the loop moving fast enough that the gravitational force would be insufficient to move the car in a circle. Some normal force would be necessary. That is, the car would remain in contact with the track. So, the car can successfully complete the loop if it starts

From *some* heights h greater than $2R$.

6. (8 points) A particle in a system has a force $\vec{F}$ exerted on it, that depends on the position x of the particle. Four possible positions of the particle are indicated on the graph. Of those four positions, which results in the system having maximum potential energy?



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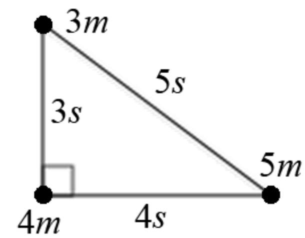
Potential energy changes are related to work done by internal forces,

$$\Delta U = -W_{\text{int}} = -\int \vec{F} \cdot d\vec{s}$$

The change in the system's potential energy, therefore, is the opposite of the area under the graph of force as a function of position. The potential energy of this system increases as the particle moves from $x = 0$, as long as the area under the curve is negative. Beyond position iii, though, the area under the curve is positive, so the potential energy of the system begins decreasing.

The system has maximum potential energy when the particle is at position iii.

7. (8 points) A system consists of three point masses with masses $3m$, $4m$, and $5m$ on the vertexes of a 3-4-5 right triangle with sides $3s$, $4s$, and $5s$, as shown. How much work must an external agent do to disassemble the system, moving the point masses to infinite separation?



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The current potential energy is the sum of the potential energy due to each pair of point masses.

$$U_{34} = -G\frac{(3m)\,(4m)}{3s} = -4Gm^2/s$$

$$U_{35} = -G\frac{(3m)\,(5m)}{3s} = -3Gm^2/s$$

$$U_{45} = -G\frac{(4m)\,(5m)}{4s} = -5Gm^2/s$$

$$U_{\text{total}} = U_{34} + U_{35} + U_{45} = -12Gm^2/s$$

The potential energy of the system is negative, as expected, since the attractive gravitational force would spontaneously bring the masses together. The potential energy of the system will be zero at infinite separation. The work done by the external agent must equal the change in potential energy from the current configuration (with negative potential energy) to infinite separation (with zero potential energy). The change from $-12Gm^2/s$ to zero is

$$+12Gm^2/s$$