
I. (16 points) A car is travelling in a straight line with velocity $+v_0$. At time $t = 0$, it begins to accelerate at $a = +k\sqrt{t}$, where k is a positive constant. Through what distance does the car travel between times $t = 0$ and $t = T$? Express this distance in terms of any or all of v_0, k, T, and physical or mathematical constants.

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The acceleration is the time-derivative of the velocity, so

$$a = dv/dt \qquad \Rightarrow \qquad v = \int a\, dt = \int k t^{1/2}\, dt = k\, \frac{t^{3/2}}{3/2} + C$$

where C is an integration constant. At time $t = 0$, the velocity is v_0, so

$$v_0 = \tfrac{2}{3}\, k \left(0^{3/2}\right) + C \qquad \Rightarrow \qquad C = v_0 \qquad \text{so} \qquad v = \tfrac{2}{3} k t^{3/2} + v_0$$

Since the velocity is the time-derivative of the position, the displacement will be the definite integral of the velocity over time.

$$v = dx/dt \qquad \Rightarrow \qquad \Delta x = \int_{t_1}^{t_2} v\, dt = \int_0^T \left(\tfrac{2}{3} k t^{3/2} + v_0\right) dt = \tfrac{2}{3}\, k\, \frac{t^{5/2}}{5/2} + v_0 t \Big|_0^T$$

So

$$\Delta x = \tfrac{4}{15}\, k\, T^{5/2} + v_0 T$$

II. (16 points) You are motorcycling home, when you find your way blocked by a swollen stream. At that instant, you are directly north of your home, and a distance $D = 19\,\mathrm{km}$ away, as illustrated. You travel a distance $a = 11\,\mathrm{km}$ along the stream, in a direction $\theta = 54°$ south of east, to a bridge. How far from your home are you now? (The bridge has negligible length, so it does not matter which side you are on.)

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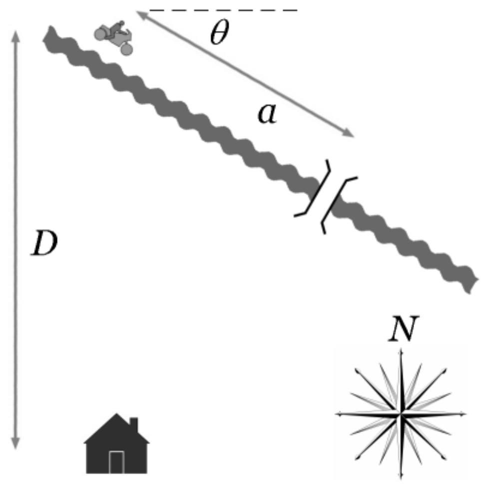

Let the vector $\vec{D}$ point from the starting position to home. Let the vector $\vec{a}$ point from the starting position to the bridge. Let the vector $\vec{b}$ point from the bridge to home. Then $\vec{a} + \vec{b} = \vec{D}$.

Looking at the East components, and noting that the East component of $\vec{D}$ is zero

$$a_E + b_E = D_E = 0$$

so

$$b_E = -a_E = -a\cos\theta = -(11\,\mathrm{km})\cos 54° = -6.47\,\mathrm{km}$$

Looking next at the North components, note that the North component of $\vec{D}$ is $-D$.

$$a_N + b_N = D_N = -D$$

So

$$b_N = -D - a_N = -D - (-a\sin\theta) = a\sin\theta - D = (11\,\mathrm{km})\sin 54° - 19\,\mathrm{km} = -10.1\,\mathrm{km}$$

The magnitude of $\vec{b}$ can be found from its components with the Pythagorean Theorem

$$\left|\vec{b}\right| = \sqrt{b_E^2 + b_N^2} = \sqrt{(-6.47\,\mathrm{km})^2 + (-10.1\,\mathrm{km})^2} = 12\,\mathrm{km}$$

1. (6 points) In what direction should you travel to get to your home from the bridge? Let the positive East axis be zero, with positive angles toward the North.

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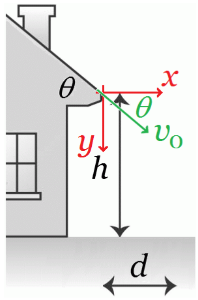
Since $|\vec{a}| < |\vec{D}|$, the diagram is qualitatively correct. The direction must be in the third quadrant, and the only choice offered in that quadrant is

$$237°$$

III. Santa Claus (not shown) steps out of the chimney and slides down the icy roof that makes an angle θ with the horizontal. He leaves the roof with a speed v_0, and lands a horizontal distance d from that point. What is the height, h, from which he fell? Express your answer in terms of parameters defined in the problem, and physical or mathematical constants. (*On Earth.*)



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This is a projectile motion problem. Choose a coordinate system. One possible system is illustrated.

The time to fall can be found from the horizontal information.

$$x_f = x_i + v_{xi}\,\Delta t + \tfrac{1}{2}a_x\,(\Delta t)^2$$

where $x_i = 0$, $x_f = d$, and $v_{xi} = v_0 \cos\theta$. The horizontal acceleration, a_x, is zero, so

$$d = 0 + v_0 \cos\theta\,\Delta t + 0 \qquad \Rightarrow \qquad \Delta t = \frac{d}{v_0 \cos\theta}$$

Looking next at the vertical information,

$$y_f = y_i + v_{yi}\,\Delta t + \tfrac{1}{2}a_y\,(\Delta t)^2$$

where $y_i = 0$, $y_f = h$, $v_{yi} = v_0 \sin\theta$, and $a_y = g$, so

$$h = 0 + v_0 \sin\theta\,\Delta t + \tfrac{1}{2}g\,(\Delta t)^2$$

Substituting the expression for Δt,

$$h = v_0 \sin\theta \left(\frac{d}{v_0 \cos\theta}\right) + \tfrac{1}{2}g\left(\frac{d}{v_0 \cos\theta}\right)^2 = d\tan\theta + \frac{gd^2}{2v_0^2\cos^2\theta}$$

2. (6 points) In the problem above, Santa Claus is traveling at an angle ϕ below the horizontal at the instant he lands on the ground. If it can be determined, how does this angle compare to the roof angle θ?

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In the absence of gravity, Santa Claus would move in a straight line, and $\phi = \theta$. But on Earth, the vertical component of his velocity increases, while the horizontal component remains constant (and non-zero). Therefore

$$90^\circ > \phi > \theta$$

3. (8 points) Mary needs to row her boat across a 100 m wide river that flows in the $-y$ direction at 1 m/s. Mary can row a a speed of 2 m/s. Assume that Mary wants to land directly across the river (i.e., in the $+x$ direction) from the point at which she started. In which direction must she point her boat with respect to the x axis?

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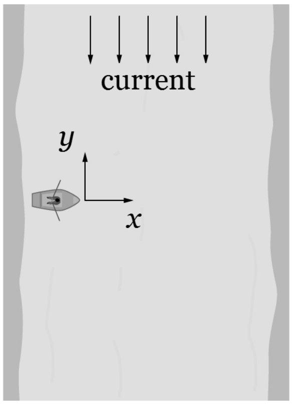
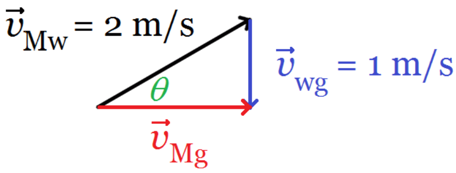


The velocity $\vec{v}_{\text{Mg}}$ of Mary with respect to the ground is

$$\vec{v}_{\text{Mg}} = \vec{v}_{\text{Mw}} + \vec{v}_{\text{wg}}$$

where $\vec{v}_{\text{Mw}}$ is the velocity of Mary with respect to the water, and $\vec{v}_{\text{wg}}$ is the velocity of the water with respect to the ground. Representing this graphically, one can see that her direction with respect to the ground is

$$\cos^{-1}(\sqrt{3}/2)$$



4. (8 points) Consider the motion diagram of an object moving at constant speed. It is initially traveling North, then turns to travel West. What is the direction, if any, of the object's acceleration when it is halfway around the turn, at the frame marked with an asterisk?

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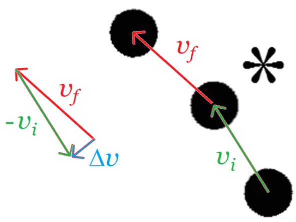
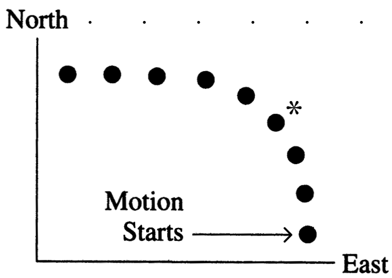
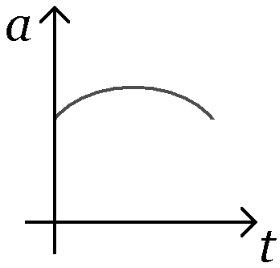




A magnified view of the region of interest is shown. Remember that the displacements (which are what the motion diagram shows) are proportional to the velocities. The change in velocity, $\Delta\vec{v} = \vec{v}_f - \vec{v}_i$, is proportional to the acceleration.

The acceleration is non-zero to the southwest.

5. (8 points) The graph shows the acceleration of an object moving in a straight line as a function of time. The velocity of the object is positive at time $t = 0$. During the time represented on the graph, the speed of this object ...



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This acceleration is always positive. Since the velocity is positive as well, the acceleration is in the same direction as the velocity and the speed of this object

increases

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6. (8 points) Consider two vectors, $\vec{A}$ and $\vec{B}$. Let their sum be $\vec{S} = \vec{A} + \vec{B}$ and their difference be $\vec{D} = \vec{A} - \vec{B}$. How must the *magnitudes* of $\vec{S}$ and $\vec{D}$ be related?

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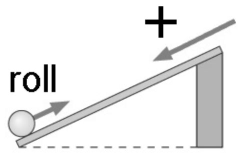
Imagine two simple cases. First, if $\vec{A}$ and $\vec{B}$ point in the same direction, the magnitude of their sum will be greater than the magnitude of their difference. On the other hand, if $\vec{A}$ and $\vec{B}$ point in opposite directions, the magnitude of their sum will be less than the magnitude of their difference. Therefore, there can be no necessary relationship between the sum and difference of two general vectors such as $\vec{A}$ and $\vec{B}$ in this question, and of the choices provided ...

None of the others is correct.

7. (8 points) The ball rolls up the ramp, then back down. The positive direction has been defined as down the ramp. Which is the appropriate graph of acceleration vs. time? (*On Earth.*)



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The ball has a constant acceleration down the slope. With that choice of coordinate system, down the slope is positive.