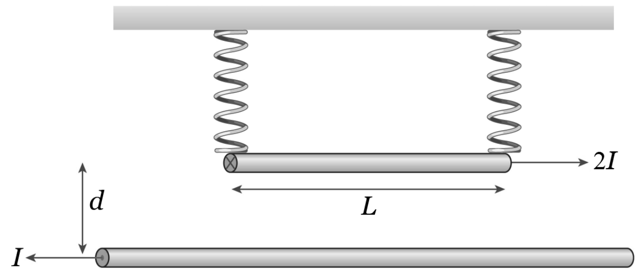
I. (16 points) A wire of length L is suspended horizontally by flexible leads above a long straight wire. Opposite currents I and $2I$ are established in the long and shorter wires, respectively, such that the shorter wire floats a distance d above the long wire with no tension in its suspension leads. If the mass of the shorter wire is m, what is the current I flowing in the long wire? Express your answer in terms of parameters defined in the problem, and physical or mathematical constants. (*On Earth, do NOT neglect gravity.*) **Note:** using a memorized formula for the force between two wires will incur a penalty for showing insufficient work.



Use Newton's Second Law. The shorter wire has an upward magnetic force F_B and a downward gravitational force mg acting on it. Let the positive x direction be upward.

$$\sum F_x = F_B - mg = ma_x = 0 \qquad \Rightarrow \qquad F_B = mg$$

The magnetic field created by the long straight wire at the location of the shorter wire can be found using Ampere's Law. Choose a circular Amperian Loop with radius d.

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{\text{thru}} \qquad \Rightarrow \qquad B 2\pi d = \mu_0 I \qquad \Rightarrow \qquad B = \frac{\mu_0 I}{2\pi d}$$

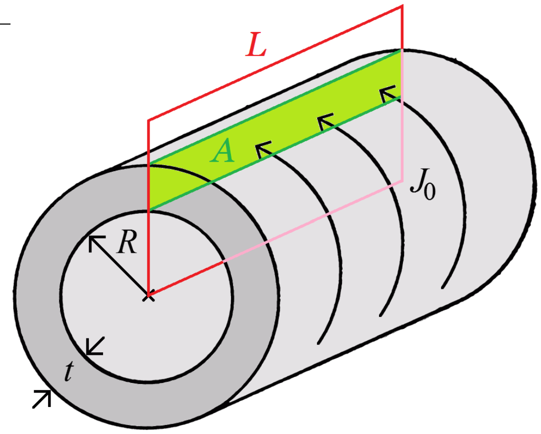
The Right-Hand-Rule shows that this magnetic field is directed into the page at the location of the shorter wire. The force on the shorter wire in this field is

$$\vec{F}_B = I_{\text{short}} \vec{L} \times \vec{B} \qquad \Rightarrow \qquad F_B = 2ILB \sin\phi$$

Note that the field is perpendicular to the length vector (which points in the direction of the current) at every point along the shorter wire, so $\phi = 90°$ and $\sin\phi = 1$. The Right-Hand-Rule confirms that the direction of the magnetic force is up the page.

$$F_B = 2ILB \sin 90° = 2IL\left(\frac{\mu_0 I}{2\pi d}\right) = mg \qquad \text{so} \qquad I = \sqrt{\frac{\pi d m g}{\mu_0 L}}$$

II. (16 points) Current flows counter-clockwise *around* an infinite hollow cylinder with inner radius R and thicknesses t, as shown. If the current density has a uniform magnitude J_0, what is the magnitude of the magnetic field in the center of the cylinder? Express your answer in terms of parameters defined in the problem, and physical or mathematical constants.



. . . .

The magnetic field in the center of the cylinder is (see MC question below) out of the page. By analogy with the ideal solenoid, the magnetic field is uniform everywhere in the hollow of the cylinder, and zero everywhere outside.

Use Ampere's Law. Choose a rectangular Amperian Loop of length L, with one side along the field line in the center of the cylinder, and the opposite side outside the cylinder (shown in red and pink).

$$\oint \vec{B}\cdot d\vec{s} = \mu_0 I_{\text{thru}} \qquad \Rightarrow \qquad \int\limits_{\text{bottom}} \vec{B}\cdot d\vec{s} + \int\limits_{\text{front}} \vec{B}\cdot d\vec{s} + \int\limits_{\text{top}} \vec{B}\cdot d\vec{s} + \int\limits_{\text{back}} \vec{B}\cdot d\vec{s} = \mu_0 I_{\text{thru}}$$

Along the top, the field is zero, so $\int\limits_{\text{top}} \vec{B}\cdot d\vec{s} = 0$. Along the front and back, $\vec{B}$ is either zero or perpendicular to $d\vec{s}$, so $\int\limits_{\text{front}} \vec{B}\cdot d\vec{s} = \int\limits_{\text{back}} \vec{B}\cdot d\vec{s} = 0$. Along the bottom of the Amperian Loop, in the center of the cylinder, the field is uniform and parallel to $d\vec{s}$, so $\int\limits_{\text{bottom}} \vec{B}\cdot d\vec{s} = BL$. So

$$\int\limits_{\text{bottom}} \vec{B}\cdot d\vec{s} + \int\limits_{\text{front}} \vec{B}\cdot d\vec{s} + \int\limits_{\text{top}} \vec{B}\cdot d\vec{s} + \int\limits_{\text{back}} \vec{B}\cdot d\vec{s} = BL + 0 + 0 + 0 = \mu_0 I_{\text{thru}}$$

The current through the Amperian Loop can be found from the current density. The current density is uniform, so

$$I_{\text{thru}} = \int \vec{J}\cdot d\vec{A} = J_0 A = J_0 L t$$

where the area Lt is marked in green. Putting these together,

$$BL = \mu_0 J_0 L t \qquad \Rightarrow \qquad B = \mu_0 J_0 t$$

where the L's canceled, as we knew they must.

1. (6 points) In the problem above, what is the direction, if any, of the magnetic field in the center of the cylinder?

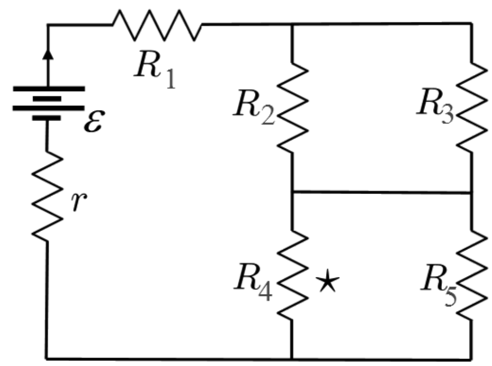
. . . .

Each "slice" of the cylinder is a loop of CCW current. Using the shortcut Right-Hand-Rule, the magnetic field in the center of the cylinder is

Out of the page.

$III.$ (16 points) The circuit shown consists of a battery of emf $\mathcal{E} = 24\,\mathrm{V}$ and internal resistance $r = 1.0\,\Omega$, and five identical resistors $R_1 \ldots R_5$, each with resistance $R = 8.0\,\Omega$. Calculate the current through the resistor R_4, indicated with a star.

. . . .



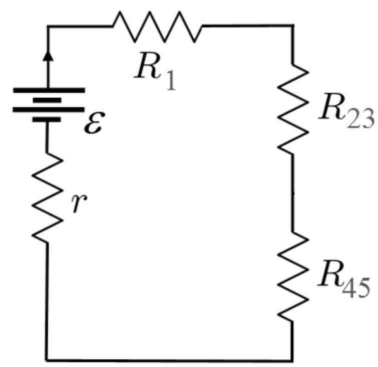
Find the equivalent resistance, so the total current supplied by the battery may be determined.

The potentials across resistors R_2 and R_3 are the same, so R_2 and R_3 are in parallel. The combination has resistance

$$R_{23} = \left(\frac{1}{R_2} + \frac{1}{R_3}\right)^{-1} = \left(\frac{1}{R} + \frac{1}{R}\right)^{-1} = R/2$$

Similarly, he potentials across resistors R_4 and R_5 are the same, so R_4 and R_5 are in parallel. That combination has resistance

$$R_{45} = \left(\frac{1}{R_4} + \frac{1}{R_5}\right)^{-1} = \left(\frac{1}{R} + \frac{1}{R}\right)^{-1} = R/2$$



The resistors r, R_1, R_{23}, and R_{45} all have the same current, and so are in series. The equivalent resistance is

$$\begin{aligned} R_{\mathrm{eq}} &= r + R_1 + R_{23} + R_{45} = r + R + \frac{R}{2} + \frac{R}{2} \\ &= r + 2R = 1.0\,\Omega + 2\,(8.0\,\Omega) = 17.0\,\Omega \end{aligned}$$

The current supplied by the battery, then, is

$$\Delta V = I_{\mathrm{eq}} R_{\mathrm{eq}} \qquad \Rightarrow \qquad I_{\mathrm{eq}} = \frac{\mathcal{E}}{R_{\mathrm{eq}}} = \frac{24\,\mathrm{V}}{17.0\,\Omega} = 1.41\,\mathrm{A}$$

That current all passes through R_1, but must split between R_2 and R_3. From symmetry considerations, it must divide evenly, and it must continue to be divided evenly as it passes through R_4 and R_5. The current through the indicated resistor R_4, then, is

$$I_4 = \frac{I_{\mathrm{eq}}}{2} = \frac{1.41\,\mathrm{A}}{2} = 0.71\,\mathrm{A}$$

2. (6 points) In the problem above, how does the power P_4 dissipated in resistor R_4 compare to that dissipated by the internal resistance, P_r?

. . . .

All the current must pass through the internal resistance r. As R_2, R_3, R_4, and R_5 are identical, symmetry considerations require that half must have come through R_4 and half must have come through R_5. So R_4 has half the current, but 8 times the resistance of r.

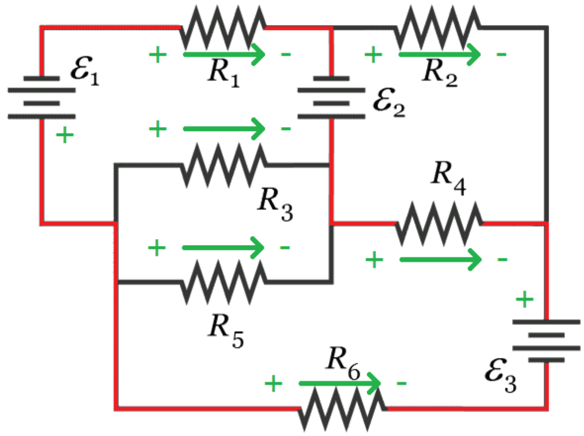
$$P_4 = I_4\,\Delta V_4 = I_4^2 R_4 = (I_r/2)^2\,(8r) = 2I_r^2 r = 2P_r$$

3. (8 points) In the illustrated circuit, current I_1 flows through resistor R_1, current I_2 flows through resistor R_2, etc. Let the positive direction of current flow be to the right through all the resistors. Which equation is a valid expression of Kirchhoff's Loop Law?

. . . .

Current directions, and high and low potential sides of the resistors and batteries have been marked in green. Among the expressions offered, only one represents a closed path (marked in red) and has correct signs for all the potential changes:

$$+\mathcal{E}_1 - I_6 R_6 + \mathcal{E}_3 + I_4 R_4 - \mathcal{E}_2 + I_1 R_1 = 0$$



4. (8 points) As an electron passes through the origin, it generates a magnetic field on the $+z$ axis that points in the $+x$ direction, as shown. In what direction could this negatively-charged particle be traveling?

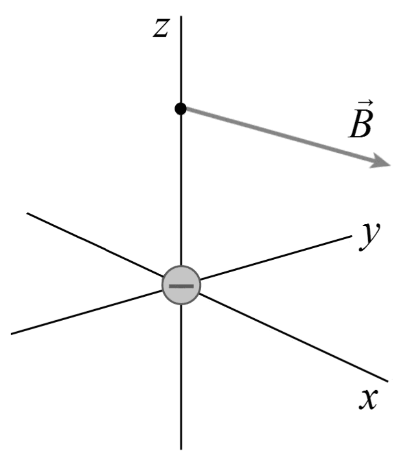
. . . .

Use the Biot-Savart Law.

$$\vec{B} = \frac{\mu_0 q}{4\pi} \frac{\vec{v} \times \hat{r}}{|\vec{r}|^2}$$

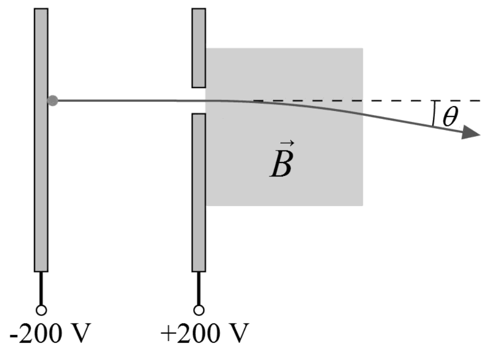
Since the particle has a negative charge, the magnetic field must be opposite the direction of $\vec{v} \times \hat{r}$. The direction opposite the magnetic field is $-\hat{x}$. The direction of $\hat{r}$ is $+\hat{z}$. As $-\hat{y} \times \hat{z} = -\hat{x}$, the particle could be traveling

In the $-y$ direction.



5. (8 points) The plates in a capacitor have the potentials indicated. A particle of unstated charge is released from rest at the -200 V plate and accelerates toward the +200 V plate. It passes through a small hole and enters a uniform magnetic field. If the particle's path is deflected down the page as shown, what is the direction of the magnetic field?

. . . .



Electric field points from high potential to low potential, and from positive charge to negative charge. Therefore, the -200 V plate is negatively charged, and the +200 V plate is positively charged. Since the particle accelerates from the -200 V plate toward the +2000 V plate, the particle must have negative charge.

The particle is moving to the right when it enters the magnetic field and experiences a magnetic force up the page. As the magnetic force on a moving charge is

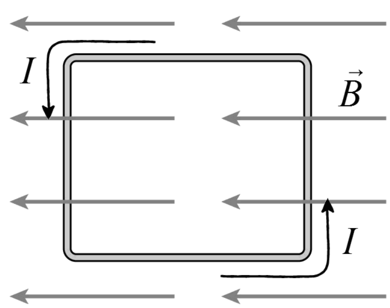
$$\vec{F} = q\vec{v} \times \vec{B}$$

the Right-Hand-Rule shows us that the magnetic field must be

Into the page.

6. (8 points) A rectangular wire loop lies in the plane of the page with counter-clockwise current I. A uniform magnetic field $\vec{B}$ is directed to the left, as shown. What is the direction of the net torque, if any, on the loop?

. . . .

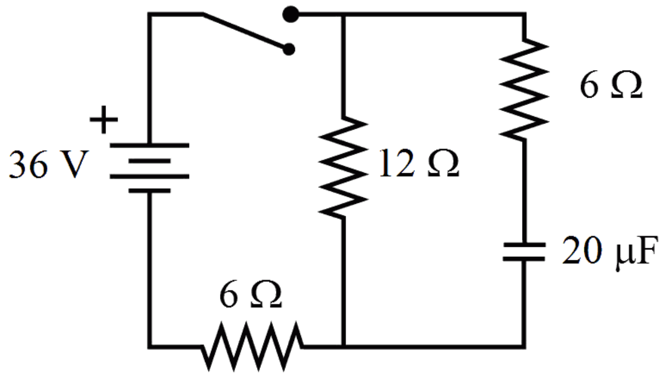


The magnetic moment of the loop points in the direction of the magnetic the current creates in the center of the loop. Using the short-cut Right-Hand-Rule, this is out of the page. As the magnetic moment is not aligned with the external field directed to the left, there will be a net torque, $\vec{\tau} = \vec{\mu} \times \vec{B}$, that tends to align the magnetic moment with the external field, and whose direction can be determined with the Right-Hand Rule.

The net torque is down the page.

7. (8 points) A circuit is constructed with a 36 V battery, a switch, two $6\,\Omega$ resistors, a $12\,\Omega$ resistor, and a $20\,\mu$F capacitor, as shown. The switch has been closed for a long time. What is the current in the $12\,\Omega$ resistor immediately upon opening the switch?

. . . .



The capacitor is fully charged after the switch has been closed "for a long time". Therefore, there is no current flowing in the right branch of the circuit. Current flows only around the left loop, passing through both the $12\,\Omega$ and bottom $6\,\Omega$ resistors. That current is

$$I_{\text{closed}} = \frac{\Delta V}{R} = \frac{36\,\text{V}}{18\,\Omega} = 2.0\,\text{A}$$

Since no current is flowing in the right branch, there is no potential difference across the right $6\,\Omega$ resistor. The potential difference across the capacitor must be the same as that across the $12\,\Omega$ resistor. That potential is

$$\Delta V_C = I_{\text{closed}} R_{\text{center}} = (2.0\,\text{A})\,(12\,\Omega) = 24\,\text{V}$$

When the switch is opened, current flows only around the right loop of the circuit, discharging the capacitor. At the first instant the switch is opened, the potential across the capacitor is the same as it was at the last instant the switch was closed. That potential is across the right $6\,\Omega$ resistor and the $12\,\Omega$ resistor, which have the same current and so are in series. That current is

$$I = \frac{\Delta V}{R} = \frac{24\,\text{V}}{18\,\Omega} = 1.3\,\text{A}$$