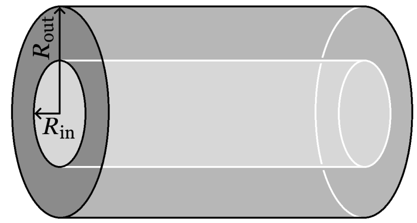

I. (16 points) A hollow conducting wire has inner radius $R_{\text{in}} = 1.8\,\text{mm}$. It carries a current whose density magnitude J varies with distance r from the central axis of the wire according to



$$J = J_0 \frac{R_{\text{in}}}{r}$$

where $J_0 = 24\,\text{A/m}^2$. If the wire carries a total current of $1.2\,\text{mA}$, what is the outer radius of the wire?

. . . .

The relationship between current and current density is

$$I = \int \vec{J} \cdot d\vec{A}$$

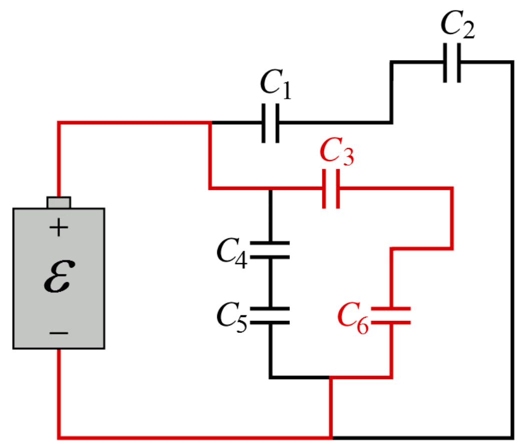
Since $\vec{J}$ varies with r, choose an area element that is small in the "r" direction. This will be a thin ring of area magnitude $dA = 2\pi r\, dr$. Note that $\vec{J}$ is parallel to this $d\vec{A}$, so

$$I = \int \vec{J} \cdot d\vec{A} = \int J\, dA = \int_{R_{\text{in}}}^{R_{\text{out}}} J_0 \frac{R_{\text{in}}}{r}\, 2\pi r\, dr = 2\pi J_0 R_{\text{in}} \int_{R_{\text{in}}}^{R_{\text{out}}} dr = 2\pi J_0 R_{\text{in}} r \Big|_{R_{\text{in}}}^{R_{\text{out}}} = 2\pi J_0 R_{\text{in}} \left(R_{\text{out}} - R_{\text{in}}\right)$$

Solve for R_{out}.

$$R_{\text{out}} = \frac{I}{2\pi J_0 R_{\text{in}}} + R_{\text{in}} = \frac{1.2\,\text{mA}}{2\pi\left(24\,\text{A/m}^2\right)\left(1.8\times10^{-3}\,\text{m}\right)} + 1.8\,\text{mm} = 4.4\,\text{mm} + 1.8\,\text{mm} = 6.2\,\text{mm}$$

II. (16 points) The circuit shown has a battery with emf (potential difference) $\mathcal{E}$, and six capacitors $C_1 \ldots C_6$ with capacitances $C_1 = 1C$, $C_2 = 2C$, $C_3 = 3C$, etc. In terms of $\mathcal{E}$, C, and physical or mathematical constants, what energy is stored in capacitor C_3?



. . . .

Capacitors C_3 and C_6 are in series (same charge).

$$C_{36} = \left(\frac{1}{C_3} + \frac{1}{C_6}\right)^{-1} = \left(\frac{1}{3C} + \frac{1}{6C}\right)^{-1}$$

$$= \left(\frac{2}{6C} + \frac{1}{6C}\right)^{-1} = \left(\frac{3}{6C}\right)^{-1} = 2C$$

There is a potential difference $\mathcal{E}$ across the combination. From the definition of capacitance,

$$= C_{36}\,\Delta V_{36} = 2C\mathcal{E} = Q_6 = Q_3$$

where $Q_3 = Q_6 = Q_{36}$ because C_3 and C_6 are in series. So the energy stored in C_3 is

$$U_3 = \frac{Q_3^2}{2C_3} = \frac{(2C\mathcal{E})^2}{2\,(3C)} = \frac{2}{3}C\mathcal{E}^2$$

1. (6 points) In the problem above, how is the potential difference ΔV_3 across capacitor C_3 related to the potential difference ΔV_6 across capacitor C_6?

. . . .

Since C_3 and C_6 are in series, they must have the same charge. From the definition of capacitance,

$$Q = C_3\,\Delta V_3 = C_6\,\Delta V_6 \qquad \Rightarrow \qquad 3C\,\Delta V_3 = 6C\,\Delta V_6 \qquad \Rightarrow \qquad \Delta V_3 = 2\Delta V_6$$

III. (16 points) The electric potential in a region of space depends on location (x, y) according to

$$V = Ax^2 - By^3$$

where $A = 4.0\,\mathrm{V/m^2}$ and $B = 3.0\,\mathrm{V/m^3}$, with respect to zero at infinity. What is the electric field at $(x, y) = (3.0, 2.0)$ m?

. . . .

Electric field and potential are related by $E_s = -\delta V/\delta s$, so

$$E_x = -\frac{\delta V}{\delta x} = \frac{-\delta}{\delta x}\left[Ax^2 - By^3\right] = -2Ax = -2\left(4.0\,\mathrm{V/m^2}\right)(3.0\,\mathrm{m}) = -24\,\mathrm{V/m}$$

and

$$E_y = -\frac{\delta V}{\delta y} = \frac{-\delta}{\delta y}\left[Ax^2 - By^3\right] = 3By^2 = 3\left(3.0\,\mathrm{V/m^3}\right)(2.0\,\mathrm{m})^2 = 36\,\mathrm{V/m}$$

Therefore

$$\vec{E} = -24\,\hat{\imath} + 36\,\hat{\jmath}\ \mathrm{V/m}$$

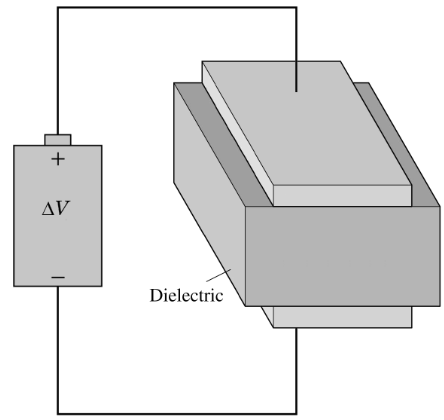
2. (6 points) If, in the problem above, you had been provided with an expression for the electric potential with respect to zero **at the origin**, instead of at infinity, how would your answer above for the electric field be affected?

. . . .

Electric field is related to the force on a probe charge, which cannot be affected by how the reference point for potential is defined.

My answer above would remain exactly the same.

3. (8 points) A parallel-plate capacitor has capacitance C with vacuum between the plates. A dielectric with dielectric constant κ is positioned to fill the space between the plates, then the capacitor is attached to a battery with potential difference ΔV, as shown. How much work must an agent or agents external to the capacitor do to remove the dielectric at constant speed?



. . . .

Use the Work-Energy Theorem

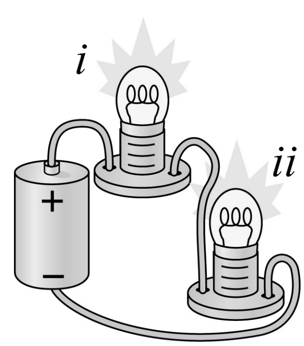
$$W_{\text{ext}} = \Delta K + \Delta U + \Delta E_{\text{th}}$$

As the dielectric is removed at constant speed, $\Delta K = 0$. There are no internal forces changing the thermal energy, so ΔE_{th}. As the capacitor is attached to a battery, the potential difference remains constant.

$$W_{\text{ext}} = \Delta U = U_f - U_i = \tfrac{1}{2}C_f\left(\Delta V\right)^2 - \tfrac{1}{2}C_i\left(\Delta V\right)^2$$

$$= \tfrac{1}{2}C\left(\Delta V\right)^2 - \tfrac{1}{2}\kappa C\left(\Delta V\right)^2 = \tfrac{1}{2}C\left(\Delta V\right)^2(1-\kappa)$$

4. (8 points) Two identical bulbs, i and ii, are attached to a battery, as shown. Because the bulbs are identical, the potential difference between the terminals of each socket (the point where the wires are attached) are the same, $\Delta V_i = \Delta V_{ii}$. If bulb ii is removed by unscrewing it from its socket, how are these two potential differences affected?



. . . .

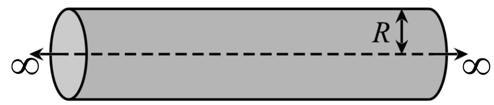
The bulbs are in series, so when bulb ii is unscrewed, no current can flow. The potential across bulb i drops to zero, so the entire potential of the battery must be across the socket for bulb ii.

ΔV_i decreases. ΔV_{ii} increases.

5. (8 points) An infinite conducting cylinder has radius R and area charge density η. Outside the cylinder, the electric potential depends on distance r from the central axis according to

$$V = \frac{-\eta}{\epsilon_0} \ln\left(\frac{r}{R}\right)$$

with respect to zero at $r = R$. With respect to that same zero point, what is the electric potential at a point $r = R/3$, inside the cylinder?



. . . .

The relationship between electric potential and field is

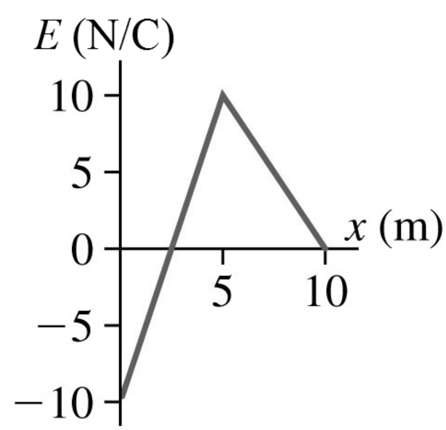
$$\Delta V = -\int \vec{E} \cdot d\vec{s}$$

As the electric field is zero inside a conductor at equilibrium, there can be no electric potential difference ΔV between any two points in or on the conducting cylinder. Every point must be at the same potential as any known point, and the potential at the surface is known.

$$V = 0$$

---

6. (8 points) An electric field depends on position x as shown. Where, in the region from $x = 0$ to $x = 10\,\mathrm{m}$ inclusive, is the electric potential greatest?



. . . .

The relationship between electric potential and field is

$$\Delta V = -\int \vec{E} \cdot d\vec{s}$$

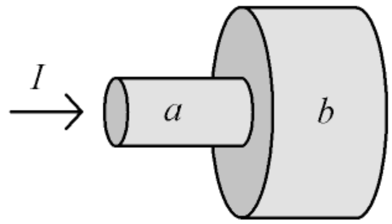
That is, the change in potential is the opposite of the area under the graph of field as a function of position.

In this case, as one moves from the origin toward $x = 2.5\,\mathrm{m}$, there is negative area under the curve, so the electric potential is increasing. As one moves beyond $x = 2.5\,\mathrm{m}$ all the way to $x = 0\,\mathrm{m}$, there is positive area under the curve, so the potential decreases. The greatest electric potential is, therefore,

At $x = 2.5\,\mathrm{m}$.

7. (8 points) Current flows from the left into the wire shown. Segment b on the right has three times the diameter of segment a on the left. The electric field magnitude in segment b on the right is one half that in segment a on the left. Compare the conductivity of segment b with that of segment a.

. . . .



The current in each segment must be the same, and can be related to the current density.

$$I_a = I_b \qquad \Rightarrow \qquad J_a A_a = J_b A_b$$

The relationship between current density and electric field, $\vec{J} = \sigma \vec{E}$, can be substituted. Letting D represent diameter,

$$\sigma_a E_a A_a = \sigma_b E_b A_b \qquad \Rightarrow \qquad \sigma_a E_a \frac{\pi}{4} D_a^2 = \sigma_b E_b \frac{\pi}{4} D_b^2$$

Solve for σ_b.

$$\sigma_b = \sigma_a \frac{E_a D_a^2}{E_b D_b^2} = \sigma_a \frac{E_a D_a^2}{(E_a/2)\,(3D_a)^2} \qquad \Rightarrow \qquad \sigma_b = (2/9)\,\sigma_a$$