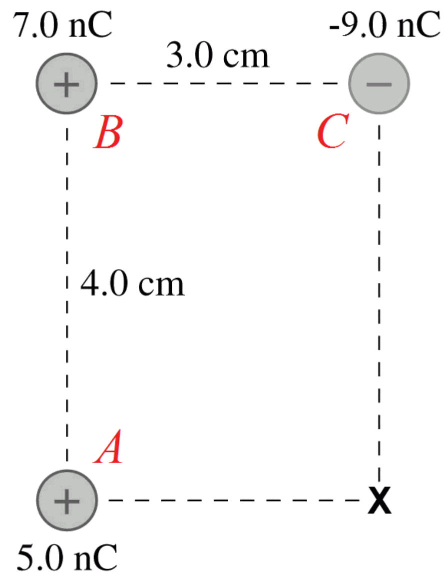
I. (16 points) A system of three charged particles is arranged on the corners of a rectangle, as shown. How much work must an external agent do move the particle with $-9.0\,\text{nC}$ of charge from rest at the upper right (where it is pictured) to rest at the lower right (marked with an "X")?



. . . .

Label the particles. Use the Work-Energy Theorem:

$$W_{\text{ext}} = \Delta K + \Delta U + \Delta E_{\text{th}}$$

Choose a system consisting of the three particles. The particles are all at rest in both their initial and final states, so $\Delta K = 0$. There are no internal non-conservative forces converting other forms of energy into thermal energy, so $\Delta E_{\text{th}} = 0$. The change in electric potential energy, ΔU, is the sum of the potential energy changes in sub-systems consisting of each of the the three pairs of particles. W_{ext} is the answer to the question. So

$$W_{\text{ext}} = 0 + \Delta U_{AB} + \Delta U_{AC} + \Delta U_{BC} + 0$$

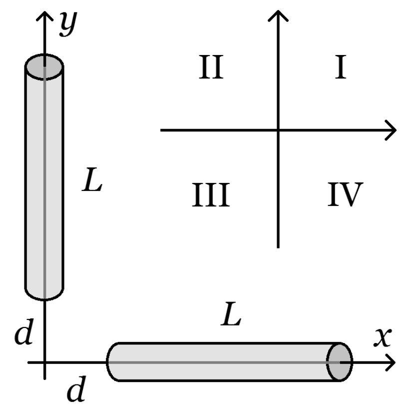
Note that the relationship between particles A and B does not change, so $\Delta U_{AB} = 0$. Then letting "i" represent initial states and "f" represent final states,

$$\begin{aligned}
W_{\text{ext}} = \Delta U_{AC} + \Delta U_{BC} &= \left(K\frac{Q_AQ_C}{R_{AC\text{f}}} - K\frac{Q_AQ_C}{R_{AC\text{i}}}\right) + \left(K\frac{Q_BQ_C}{R_{BC\text{f}}} - K\frac{Q_BQ_C}{R_{BC\text{i}}}\right) \\
&= KQ_C\left[Q_A\left(\frac{1}{R_{AC\text{f}}} - \frac{1}{R_{AC\text{i}}}\right) + Q_B\left(\frac{1}{R_{BC\text{f}}} - \frac{1}{R_{BC\text{i}}}\right)\right] \\
&= \left(8.988\times10^{9}\,\text{N}\cdot\text{m}^2/\text{C}^2\right)\left(-9.0\times10^{-9}\,\text{C}\right)\left[5.0\times10^{-9}\,\text{C}\left(\frac{1}{3.0\times10^{-2}\,\text{m}} - \frac{1}{5.0\times10^{-2}\,\text{m}}\right)\right. \\
&\qquad \left. + 7.0\times10^{-9}\,\text{C}\left(\frac{1}{5.0\times10^{-2}\,\text{m}} - \frac{1}{3.0\times10^{-2}\,\text{m}}\right)\right] \\
&= 2.2\times10^{-6}\,\text{J}
\end{aligned}$$

1. (6 points) Two identical, thin, insulating rods with length L are placed on the x and y axes at $x = d$ and $y = d$, as shown. Their linear charge densities λ depend on position according to

$$\text{rod on } x: \quad \lambda_x = \lambda_0 \frac{x}{L} \qquad\qquad \text{rod on } y: \quad \lambda_y = \lambda_0 \frac{y}{L}$$

where λ_0 is a constant. What is the direction of the electric potential at the origin?



Electric potential is a scalar!

No direction.

II. (16 points) In the problem above, what is the electric potential at the origin, with respect to zero at infinite distance? Express your answer in terms of parameters defined in the problem, and physical or mathematical constants.

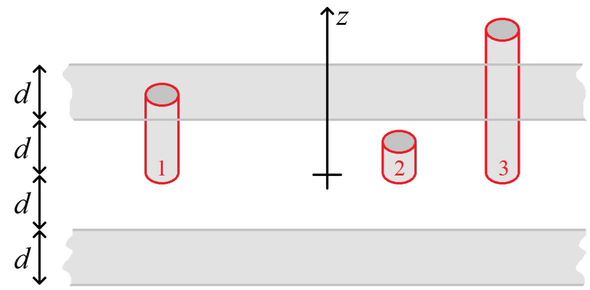
Because the rods are identical and placed the same distance from the origin, the electric potential there due to both rods is twice that due to one rod. Consider the rod on the x axis. Break it into point-like bits of width dx. Each has a bit of charge $dq = \lambda\, dx$. So

$$V_{\text{both}} = 2V_{\text{one}} = 2\int dV = 2\int \frac{K\, dq}{r} = 2\int \frac{K\lambda\, dx}{x}$$

$$= 2K \int_d^{d+L} \frac{\lambda_0\,(x/L)\, dx}{x} = \frac{2K\lambda_0}{L} \int_d^{d+L} dx = \frac{2K\lambda_0}{L} x \Big|_d^{d+L} = \frac{2K\lambda_0}{L}\Big[(d+L) - d\Big] = 2K\lambda_0$$

III. (16 points) Two identical infinite insulating slabs have thickness d and uniform volume charge density ρ. They are separated by a distance $2d$, as shown. Find an expression for the electric field magnitude within the top slab as a function of distance z from the center of the gap (that is, for $d < z < 2d$), in terms of parameters defined in the problem, and physical or mathematical constants.

. . . .



Use Gauss' Law, $\epsilon_0 \Phi = q_{\text{in}}$. The field at $z = 0$ must be zero, due to symmetry. Choose a Gaussian Surface like #1, on the left, with end-cap area A. Again, due to symmetry, there can be no flux through the curved side, so

$$\Phi = \oint \vec{E} \cdot d\vec{A} = EA$$

The charge inside is the density times the volume, but charge doesn't fill the entire volume—only the part where $z > d$. So

$$q_{\text{in}} = \rho V = \rho A (z - d)$$

Putting these together,

$$\epsilon_0 EA = \rho A (z - d) \qquad \Rightarrow \qquad E = \rho (z - d) / \epsilon_0$$

2. (6 points) In the problem above, describe the electric field magnitude between the slabs ($-d < z < d$). Describe the field magnitude beyond the slabs ($|z| > 2d$).

. . . .

Consider Gaussian Surface #2 above. There is no charge inside it, so no flux through it, and no field between the slabs. Consider Gaussian Surface #3 above. No matter how much taller than $2d$ it is, there is always the same amount of charge inside it. The flux through it does not depend on z, and neither does the field at its top end.

Field magnitude is zero between the slabs. It is a non-zero constant beyond the slabs.

3. (8 points) A positively charged particle lies outside a finite cylinder, on its axis, as shown. Compare the electric fluxes through the left (Φ_L) and right (Φ_R) end caps of the cylinder.

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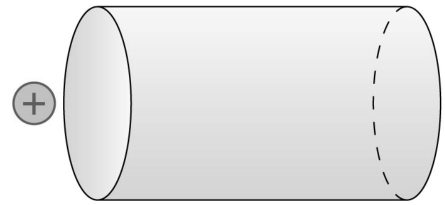
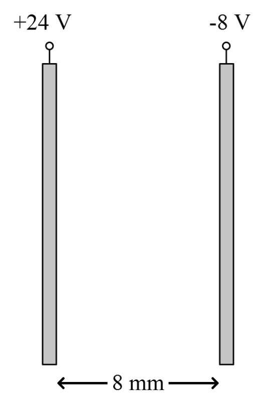


Field lines only enter the left end cap. Field lines only exit the right end cap. The fluxes at each end cap must, therefore, have opposite signs.

Every field line that exits the right end cap must have entered the left end cap. But not all field lines that enter the left end cap will exit the right end cap—some exit the curved side of the cylinder.

Φ_L and Φ_R have opposite signs, with $|\Phi_R| < |\Phi_L|$.

4. (8 points) The plates of an ideal parallel-plate capacitor are 8.0 mm apart. The negative plate is at -8.0 V, while the positive plate is at +24 V. What is the electric potential inside the capacitor, at a point 2.0 mm from the negative plate?



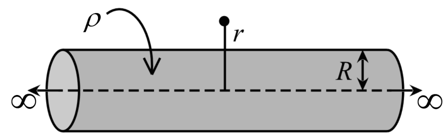
. . . .

Electric potential increases linearly from the negative to positive plate in an ideal parallel-plate capacitor. There's a 32 V increase in this capacitor, from -8 V to +24 V. With a separation of 8 mm, that's an increase of 4 V/mm.

In 2 mm, then, the potential increases 8 V. 8 V greater than the -8 V at the negative plate is

0 V

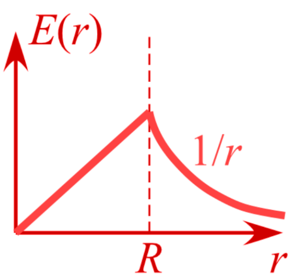
5. (8 points) An infinite insulating cylinder has radius R and uniform volume charge density ρ. Which graph below best represents the magnitude of the electric field $E(r)$ as a function of the distance r from the cylinder axis?



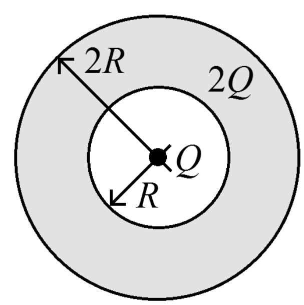
A Gaussian Surface inside the cylinder contains charge, so there's a non-zero flux, and the field cannot be zero everywhere inside the cylinder. That excludes two choices offered. But symmetry considerations require that the field be zero on the cylinder axis. That excludes another of the choices offered.

The field outside the cylinder cannot decrease linearly with distance, as it would eventually go to zero at a finite distance. That excludes another of the choices offered.

The only remaining choice is:



6. (8 points) A hollow conducting sphere has inner radius R and outer radius $2R$. It carries a charge $2Q$. Centered within it lies a particle with charge Q. Compare the area charge densities on the inner (η_{in}) and outer (η_{out}) surfaces of the sphere.



The field inside the thickness of a conductor at equilibrium is zero. So, the flux through a Gaussian Surface lying in that thickness is zero, and the net charge within the Gaussian Surface is zero.

There must, therefore, but a net charge of $-Q$ on the inner surface of the hollow conducting sphere, to balance the Q charge on the particle. Since charge is conserved, there must be a net charge of $3Q$ on the outer surface of the hollow conducting sphere. The charge, and therefore the area charge density, of the inner and outer surfaces must be opposite.

The outer surface of the hollow conducting sphere has three times the charge magnitude that the inner surface does. But because the area of a sphere is proportional to the square of its radius ($A = 4\pi r^2$), the outer surface has four times the area that the inner surface does.

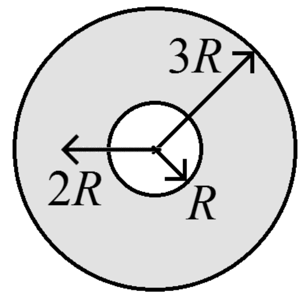
$|\eta_{\text{out}}| < |\eta_{\text{in}}|$. η_{in} and η_{out} have opposite signs.

7. (8 points) A hollow insulating sphere has inner radius R, outer radius $3R$, and uniform volume charge density ρ. If Gauss' Law is used to find the magnitude of the electric field E at a distance $2R$ from the center, the Gaussian Surface may be chosen so

$$\epsilon_0 \oint \vec{E} \cdot d\vec{A} = q_{\text{in}} \qquad \Rightarrow \qquad \epsilon_0 E A = \rho V \qquad \Rightarrow \qquad E = \frac{\rho V}{\epsilon_0 A}$$

With this approach, what is A? What is V?

. . . .



A is the area of the Gaussian Surface, which is a sphere of radius $2R$. V is the volume of charge within the Gaussian Surface, which is a sphere of radius $2R$, less the hollow sphere of radius R. That's

$$V = \frac{4}{3}\pi\left(2R\right)^3 - \frac{4}{3}\pi\left(R\right)^3 = \frac{4}{3}\pi\left(8-1\right)R^3$$

So

$$A = 4\pi\left(2R\right)^2 \qquad \text{and} \qquad V = \frac{4}{3}\pi\, 7R^3$$