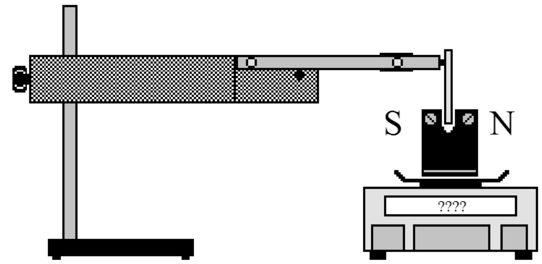
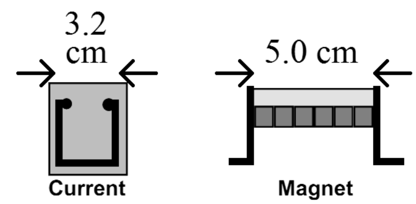
I. (16 points) A magnet with a length of 5.0 cm has a uniform 11.76 mT field between its poles. It is placed on an electronic balance, with the North pole to the right and the South pole to the left, and the balance indicates 160.89 g. Then a wire 3.2 cm long is placed between the poles, and a current of 4.50 A flows into the page. What is the reading on the balance while the current is flowing? (*On Earth.*)





The force on a current-carrying wire in a magnetic field is

$$\vec{F} = I\vec{\ell} \times \vec{B}$$

The magnetic field points from North to South, which is right to left on the page. The current is into the page, so the direction of the force, by the Right Hand Rule, is upward. Note that this is the force *on the wire*, so the force on the magnet, by Newton's Third Law, is downward. The reading on the balance will be *greater* than 160.89 g.

The magnitude of the force on the wire (and also on the magnet) is

$$F = I\ell B \sin\phi = (4.50\,\mathrm{A})\left(3.2 \times 10^{-2}\,\mathrm{m}\right)\left(11.76 \times 10^{-3}\,\mathrm{T}\right)\sin 90^\circ = 1.69 \times 10^{-3}\,\mathrm{N}$$

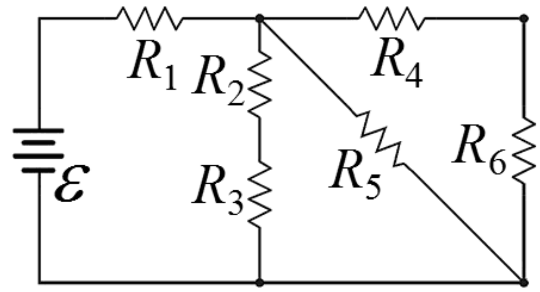
But the balance is calibrated in grams. The mass that would have this gravitational force on Earth is

$$m = \frac{F}{g} = \frac{1.69 \times 10^{-3}\,\mathrm{N}}{9.8\,\mathrm{N/kg}} = 0.173 \times 10^{-3}\,\mathrm{kg} = 0.173\,\mathrm{g}$$

Adding this to the reading already on the balance, due to the magnet, results in a reading of

$$160.89\,\mathrm{g} + 0.173\,\mathrm{g} = 161.06\,\mathrm{g}$$

II. (16 points) The battery in the circuit shown has emf $\mathcal{E}$. All six resistors have identical resistance R. What is the current through resistor R_5? Express your answer in terms of parameters defined in the problem, and physical or mathematical constants.



. . . .

Combine resistances in series or parallel until the equivalent resistance is determined, then work back to find the desired current.

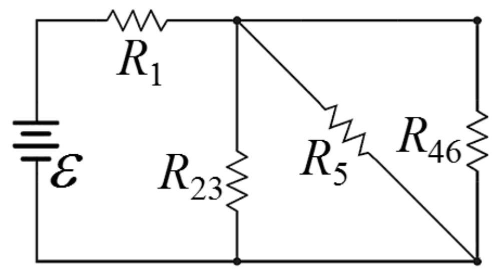
Resistances R_2 and R_3 are in series, as are R_4 and R_6.

$$R_{23} = R_2 + R_3 = R + R = 2R \qquad \text{and} \qquad R_{46} = R_4 + R_6 = R + R = 2R$$

Resistances R_{23}, R_5, and R_{46} are in parallel.

$$R_{23456} = \left(\frac{1}{R_{23}} + \frac{1}{R_5} + \frac{1}{R_{46}}\right)^{-1} = \left(\frac{1}{2R} + \frac{1}{R} + \frac{1}{2R}\right)^{-1} = \left(\frac{4}{2R}\right)^{-1} = \frac{R}{2}$$

Resistances R_1 and R_{23456} are in series.

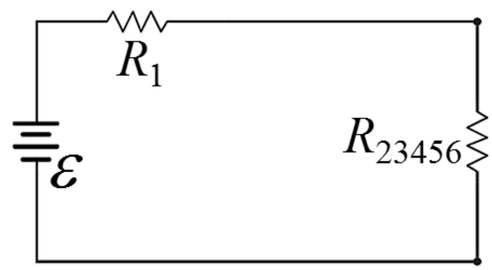


$$R_{123456} = R_1 + R_{23456} = R + R/2 = 3R/2$$

From the definition of resistance, the current supplied by the battery is

$$I_{123456} = \frac{\Delta V_{123456}}{R_{123456}} = \frac{\mathcal{E}}{3R/2} = \frac{2\mathcal{E}}{3R}$$

As resistances R_1 and R_{23456} were in series,

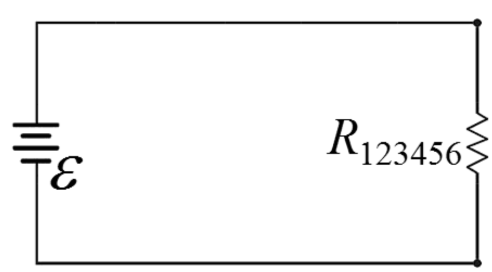


$$I_{123456} = I_1 = I_{23456} = \frac{2\mathcal{E}}{3R}$$

The potential across R_{23456} is

$$\Delta V_{23456} = I_{23456} R_{23456} = \left(\frac{2\mathcal{E}}{3R}\right)\left(\frac{R}{2}\right) = \frac{\mathcal{E}}{3}$$

As resistances R_{23}, R_5, and R_{46} were in parallel,



$$\Delta V_{23456} = \Delta V_{23} = \Delta V_5 = \Delta V_{46} = \frac{\mathcal{E}}{3}$$

So

$$I_5 = \frac{\Delta V_5}{R_5} = \frac{\mathcal{E}/3}{R} = \frac{\mathcal{E}}{3R}$$

1. (6 points) Let the power dissipated in resistor R_5 in the problem above be P_5. If the emf of the battery were doubled, and the resistance of all the resistors was halved, what power P_5' would now be dissipated in resistor R_5?

. . . .

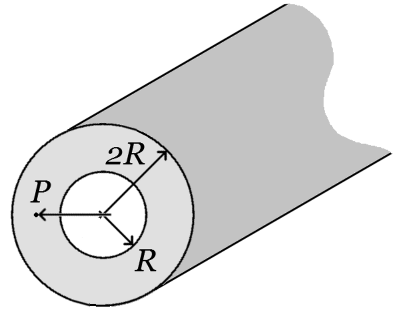
The power dissipated in a resistor is $P = I\,\Delta V = (\Delta V)^2/R$. For the dimensions to be correct, then, the power dissipated in every resistor in this circuit must be proportional to $\mathcal{E}^2/R$. With the emf doubled and the resistances halved,

$$P_5' = 8P_5$$

III. (16 points) An infinite straight hollow wire has inner radius R and outer radius $2R$, as illustrated. Its current density, $\vec{J}$, is directed out of the page and has a magnitude that varies with distance r from the center according to

$$J = J_0 \frac{R}{r}$$

where J_0 is a positive constant. Find the magnetic field magnitude at a point P which is a distance $3R/2$ from the center, in terms of parameters defined in the problem, and physical or mathematical constants.



. . . .

Use Ampere's Law, choosing a circular path passing through the point P and centered on the cylinder axis. This choice gives $\vec{B}$ a constant magnitude, and a direction that is everywhere parallel to the path element $\vec{ds}$.

$$\oint \vec{B} \cdot \vec{ds} = \mu_0 I_{\text{thru}} \qquad \Rightarrow \qquad B \oint ds = \mu_0 \int \vec{J} \cdot \vec{dA}$$

Let the area element $\vec{dA}$ be a thin ring with a direction parallel to the current density $\vec{J}$. Note that $\oint ds$ is the circumference of the circle constituting the path.

$$B 2\pi \left(\frac{3R}{2}\right) = \mu_0 \int_R^{3R/2} J_0 \frac{R}{r}\, 2\pi r\, dr = \mu_0 J_0 2\pi R \int_R^{3R/2} dr = \mu_0 J_0 2\pi R\, r \Big|_R^{3R/2} = \mu_0 J_0 2\pi R \left(\frac{3R}{2} - \frac{R}{2}\right) = \mu_0 J_0 \pi R^2$$

So

$$B = \frac{\mu_0 J_0 R}{3}$$

2. (6 points) In the problem above, what is the direction of the magnetic field at the illustrated point P?

. . . .

Using either the Biot-Savart Law for current

$$\vec{dB} = \frac{\mu_0 I}{4\pi} \frac{\vec{d\ell} \times \hat{r}}{|\vec{r}|^2}$$

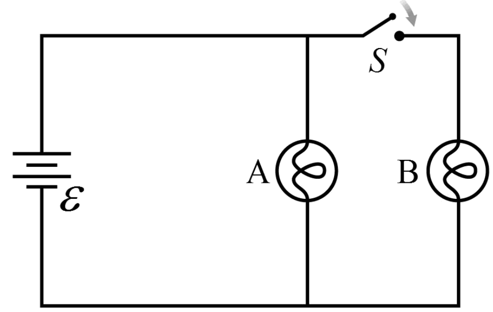
with $\vec{d\ell}$ out the page and $\hat{r}$ to the left, or the equivalent Right Hand Rule shortcut with thumb pointed out of the page and curled fingers indicating the magnetic field direction, one finds that the field at P is

Down the page.

3. (8 points) The circuit shown has identical light bulbs A and B, switch S, and a battery with emf $\mathcal{E}$. Bulb A glows when switch S is open. When the switch is closed, the brightness of bulb A ...

. . . .

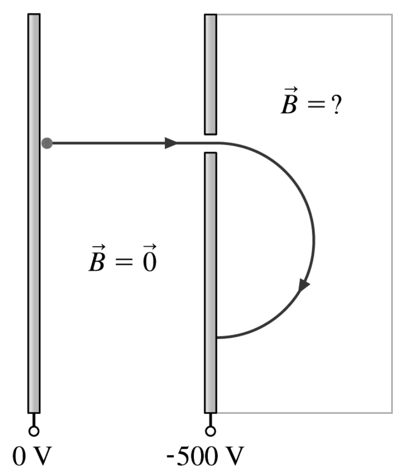
If the battery is ideal, the potential across it is always the same as its emf. The potential across bulb A will not be affected by the closing of switch S, so the brightness remains the same. But if the battery is real, then the potential across it depends on the current through it, due to the potential drop across its internal resistance. The equivalent resistance of the circuit decreases when switch S is closed, so more current passes through the battery. The potential drop across its internal resistance increases, so the potential across its terminals (and therefore across bulb A) decreases, dimming bulb A.



So, the brightness of bulb A ...

remains the same if the battery is ideal, but decreases if the battery is real.

4. (8 points) A charged particle is released from rest at a plate with an electric potential of 0 V. It accelerates toward a plate with an electric potential of -500 V, and passes through a small hole to enter a region of uniform magnetic field. If the path of the particle curves as shown, what is the direction of the magnetic field?



. . . .

Electric field points from high to low potential, so is to the right between the plates. As the electric force is $\vec{F}_E = q\vec{E}$, a particle that accelerates to the right from rest must have positive charge.

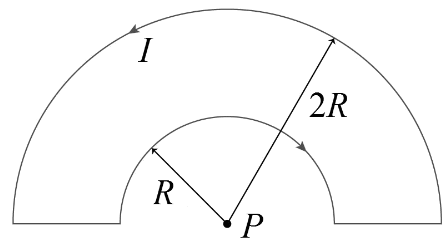
The magnetic force on this particle is down the page when it enters the magnetic field. This force is

$$\vec{F}_B = q\vec{v} \times \vec{B}$$

Since q is positive, $\vec{v} \times \vec{B}$ must be down the page. With $\vec{v}$ rightward, by the Right Hand Rule, the magnetic field must be

Out of the page.

5. (8 points) Current I flows counter-clockwise around the loop made of two connected arcs with radii R and $2R$, as shown. What is the magnetic field at the center of the arcs, P?



. . . .

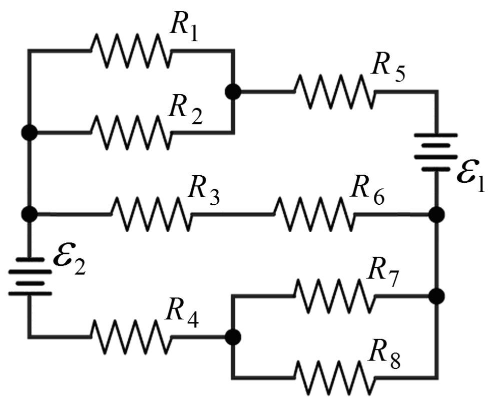
Use the Biot-Savart Law. The current in the straight pieces is directly toward or away from P, so contributes no field. In the arcs,

$$\vec{B} = \int \frac{\mu_0 I}{4\pi} \frac{d\vec{\ell} \times \hat{r}}{|\vec{r}|^2} \qquad \Rightarrow \qquad B = \frac{\mu_0 I}{4\pi r^2}(\pi r) = \frac{\mu_0 I}{4r}$$

The inner arc has radius $r = R$, so contributes field magnitude $\mu_0 I/4R$, directed into the page by the Right Hand Rule. The outer arc has radius $r = 2R$, so contributes field magnitude $\mu_0 I/8R$, directed out of the page by the Right Hand Rule. The total field at point P, then, is

$\frac{\mu_0 I}{8R}$ in to the page

6. (8 points) Resistors R_1 through R_8 in the illustrated circuit carry currents I_1 through I_8, respectively. Let the positive direction of current flow be left to right through all the resistors. Which equation is a valid expression of Kirchhoff's Loop Law?

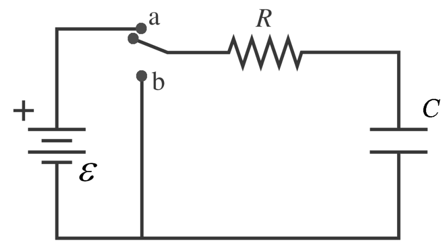


. . . .

With the positive direction of current flow defined as left to right, traversing a resistor from left to right should be represented as a *decrease* in potential, $-IR$. Check each possible answer starting with $+\mathcal{E}_1$. If an incorrect potential change is encountered, or the answer does not represent a closed loop, that answer must be rejected. The only option presented which represents a closed loop and has all potential changes correct is

$$+\mathcal{E}_1 + I_5 R_5 + I_1 R_1 + \mathcal{E}_2 - I_4 R_4 - I_7 R_7 = 0$$

7. (8 points) The switch in the illustrated circuit is set to position "b" for a long time, then set to position "a" for a time t_a, then set back to position "b". What is the potential across the resistor at the instant the switch returns to "b"?



. . . .

Charge on the capacitor remains constant as the switch is thrown from "a" to "b". At the end of time t_a, that charge is

$$Q(t_a) = Q_\infty \left(1 - e^{t_a/RC}\right) = C\mathcal{E}\left(1 - e^{t_a/RC}\right)$$

So

$$\Delta V_C(t_a) = Q(t_a)/C = \mathcal{E}\left(1 - e^{t_a/RC}\right)$$

Once the switch is thrown back to "b", the potential difference across the resistor must be the same as the potential difference across the capacitor.

$$\Delta V_R = \Delta V_C(t_a) = \mathcal{E}\left(1 - e^{-t_a/RC}\right)$$