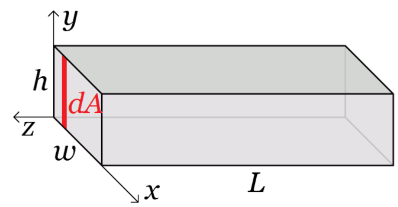

I. (16 points) A wire is rectangular in cross-section, with height h and width w. A segment of length L lies along the $-z$ axis, as shown. The current density in the wire is non-uniform, depending on position x according to

$$\vec{J} = -J_0 \left(\frac{x}{w}\right) \hat{z}$$



where J_0 is a positive constant. What magnitude current flows through the wire? Express your answer in terms of parameters defined in the problem, and physical or mathematical constants.

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Current and current density are related by

$$I = \int \vec{J} \cdot d\vec{A}$$

The current density varies with x. Choose an element of area that is small in that direction, and whose direction is parallel to $\vec{J}$. This is $d\vec{A} = -h\,dx\,\hat{z}$. Add up (integrate) the current through all the bits of area from $x = 0$ to $x = w$.

$$I = \int J \cos\theta dA = \int_0^w J_0 \left(\frac{x}{w}\right) \cos 0^\circ \, h\, dx = \frac{J_0 h}{w} \int_0^w x\, dx = \frac{J_0 h}{w} \left[\frac{x^2}{2}\right]_0^w = \frac{J_0 h}{2w}\left(w^2 - 0^2\right) = \frac{J_0 hw}{2}$$

II. (16 points) The battery in the circuit shown has emf (potential difference) $\mathcal{E}$. All seven capacitors have identical capacitances C. What is the charge on capacitor C_7? Express your answer in terms of parameters defined in the problem, and physical or mathematical constants.

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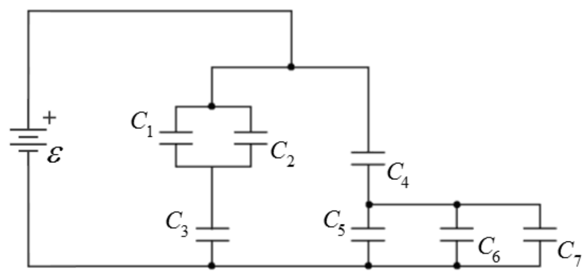
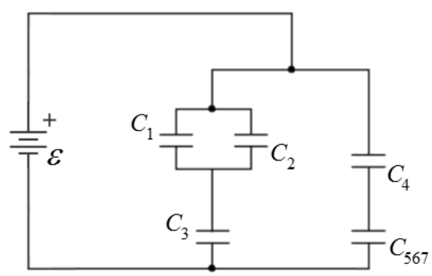


Combine capacitors in series or parallel until a charge on and potential across an equivalent capacitance can be determined. Then work back, finding charges and potentials on individual capacitors.

Start with capacitors 5, 6, and 7. They are in parallel (same potential).

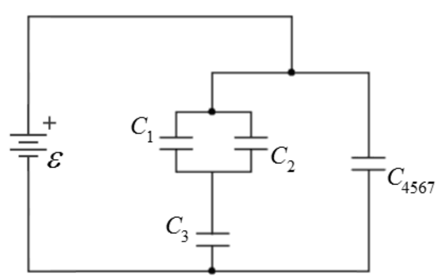
$$C_{567} = C_5 + C_6 + C_7 = 3C$$



Capacitor C_4 is in series with capacitor C_{567} (same charge).

$$C_{4567} = \left(\frac{1}{C_4} + \frac{1}{C_{567}}\right)^{-1} = \left(\frac{1}{C} + \frac{1}{3C}\right)^{-1}$$

$$= \left(\frac{3}{3C} + \frac{1}{3C}\right)^{-1} = \left(\frac{4}{3C}\right)^{-1} = \frac{3C}{4}$$



The potential across C_{4567} is $\mathcal{E}$, so the charge on C_{4567} is

$$Q_{4567} = C_{4567}\,\Delta V_{4567} = \left(\frac{3C}{4}\right)\mathcal{E} = \frac{3C\mathcal{E}}{4}$$

Because C_4 and C_{567} are in series,

$$Q_{567} = Q_4 = Q_{4567} = 3C\mathcal{E}/4$$

The potential across C_{567} is

$$\Delta V_{567} = \frac{Q_{567}}{C_{567}} = \frac{3C\mathcal{E}/4}{3C} = \mathcal{E}/4$$

Because capacitors 5, 6, and 7 are in parallel,

$$\Delta V_7 = \Delta V_6 = \Delta V_5 = \Delta V_{567} = \mathcal{E}/4$$

So

$$Q_7 = C_7\,\Delta V_7 = C\mathcal{E}/4$$

1. (6 points) Let your answer to the problem above be Q_7. If the emf of the battery were doubled, and the capacitance of all the capacitors was halved, what charge Q_7' would be on capacitor C_7?

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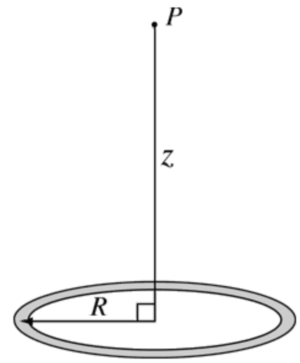
From the definition of capacitance, $Q = C\,\Delta V$, to be dimensionally correct, the charge on C_7 must be proportional to the product $C\mathcal{E}$, so

$$Q_7' = Q_7$$

III. (16 points) The electric potential on the axis of a uniform ring of charge, with respect to zero at infinity, is



$$V = K\frac{Q}{\sqrt{z^2 + R^2}}$$

where Q is the charge on the ring, R is the radius of the ring, and z is the distance along the axis from the center of the ring. From this potential, derive an expression for the magnitude of the electric field on the axis of the ring, as a function of distance z, and in terms of other parameters defined in the problem, and physical or mathematical constants.

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From symmetry considerations, the electric field must be directed along the z axis, so $E = E_z$. Using the relationship between field and potential,

$$E = E_z = -\frac{\delta V}{\delta z} = \frac{-\delta}{\delta z}\left(\frac{KQ}{\sqrt{z^2 + R^2}}\right)$$

$$= -KQ\frac{\delta}{\delta z}\left(z^2 + R^2\right)^{-1/2} = -KQ\left(-\frac{1}{2}\right)\left(z^2 + R^2\right)^{-3/2}(2z) = \frac{KQz}{\left(z^2 + R^2\right)^{3/2}}$$

2. (6 points) If, in the problem above, you had been provided with an expression for the electric potential on the axis of a uniform ring of charge with respect to zero at **the center**, how would your expression above for the magnitude of the electric field be affected?

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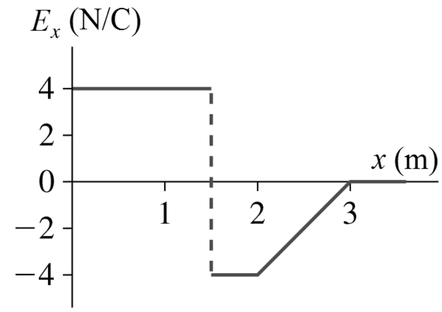
The potential difference between infinity and the center of the ring is some value, a constant. Changing the reference point for the potential would change the potential by that constant. That is, the potential on the axis would be

$$V = K\frac{Q}{\sqrt{z^2 + R^2}} + C$$

where C is a constant. After taking the derivative with respect to z,

My expression above would remain exactly the same.

3. (8 points) An electric field depends on position x as shown. Where, in the region from $x = 0$ to $x = 3\,\text{m}$ inclusive, is the electric potential greatest?



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The relationship between electric potential and field is

$$\Delta V = -\int \vec{E} \cdot d\vec{s}$$

That is, the change in potential is the opposite of the area under the graph of field as a function of position.

In this case, as one moves from the origin toward $x = 1.5\,\text{m}$, there is positive area under the curve, so the electric potential is decreasing. As one moves beyond $x = 1.5\,\text{m}$ toward $x = 3\,\text{m}$, there is negative area under the curve, so the potential begins to increase again. But by the time $x = 3\,\text{m}$ is reached, there has been less "negative area" than "positive area", so the potential is still less than it's original value. The greatest electric potential is

At $x = 0\,\text{m}$.

4. (8 points) Three identical ideal parallel-plate capacitors are attached to three identical batteries. All three look like system i. While the capacitors remain attached to their batteries, insulating handles are used to double the distance between the plates in system ii, and to stretch the plates in system iii to double their original area. Compare the potential energies stored in each capacitor.

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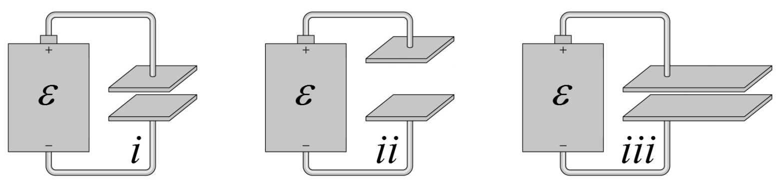


The potential energy stored in a capacitor, with respect to zero in the uncharged state, is

$$U = \tfrac{1}{2} C \left(\Delta V\right)^2 = \frac{Q^2}{2C}$$

As the capacitors remain connected to their batteries, the potential across each capacitor remains constant, making $U = \frac{1}{2} C \left(\Delta V\right)^2$ more useful.

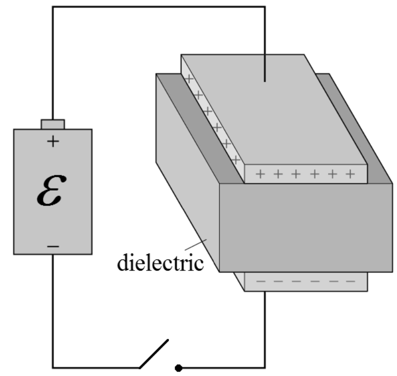
The capacitance of an ideal parallel-plate capacitor is

$$C = \epsilon_0 \frac{A}{d}$$

so the capacitor with greater area A will have more energy stored, and the capacitor with greater plate separation d will have less energy stored. The ranking is

$$iii > i > ii$$

5. (8 points) When the capacitor shown has no dielectric between the plates, its capacitance is C_0. An insulating slab with dielectric constant κ is inserted to fill the space between the plates. The capacitor is then connected to a battery with emf (or potential difference) $\mathcal{E}$, allowed to fully charge, then disconnected. How much work, if any, must be done by an agent external to the capacitor to remove the insulating slab?



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As the capacitor is disconnected from the battery when the external agent is removing the dielectric, the charge on the plates can't change. That charge is

$$Q = C\,\Delta V = \kappa C_0 \mathcal{E}$$

The work done by the external agent can be found using the Work-Energy Theorem

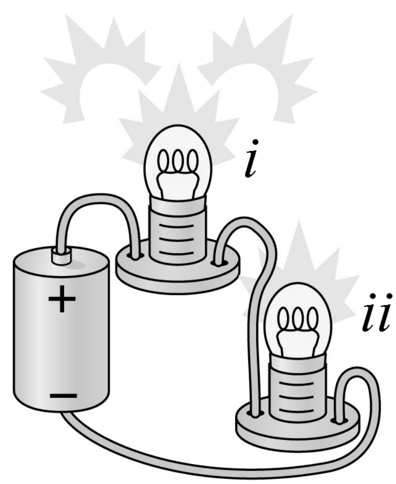
$$W_{\text{ext}} = \Delta K + \Delta U + \Delta E_{\text{th}}$$

The external agent needn't change the kinetic energy, and there are no internal non-conservative forces to change the thermal energy of the system. The potential energy stored in a capacitor, with respect to zero in the uncharged state, is $U = \frac{1}{2}C\left(\Delta V\right)^2 = \dfrac{Q^2}{2C}$. As the charge on the capacitor remains constant, $U = \frac{Q^2}{2C}$ is useful. So

$$W_{\text{ext}} = U_f - U_i = \frac{Q^2}{2C_f} - \frac{Q^2}{2C_i} = \frac{Q^2}{2C_0} - \frac{Q^2}{2\kappa C_0} = \frac{Q^2}{2C_0}\left(1 - \frac{1}{\kappa}\right) = \frac{\left(\kappa C_0 \mathcal{E}\right)^2}{2C_0}\left(1 - \frac{1}{\kappa}\right) = \frac{1}{2}\kappa C_0 \mathcal{E}^2\left(\kappa - 1\right)$$

which is positive, as expected.

6. (8 points) Bulbs i and ii are attached to a battery, as shown. The bulbs are **NOT** identical, and bulb i is brighter than bulb ii. If the polarity of the battery was reversed, so bulb ii was now attached to the positive terminal, compare the resulting bulb brightnesses and the currents though each bulb.



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Reversing the polarity of the battery will change the direction current flows through the circuit, but does not affect the equivalent resistance of the circuit, so the current supplied by the battery does not change. The bulbs are in series, so that unchanged current passes through each bulb. With the same current flowing through the same bulbs, the brightnesses will be unchanged.

Bulb i is still brighter. The current through bulb i is **the same as** the current through bulb ii.

7. (8 points) A wire of uniform diameter is made by joining two different pieces, each with its own conductivity. Current I_1 flows into piece 1, which has conductivity σ_1. Current I_2 flows out of piece 2, which has conductivity σ_2. Under what circumstances, if any, will there be a layer of **positive** charge at the junction between the pieces?

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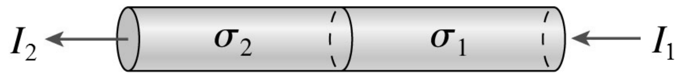

Consider a Gaussian Surface that encloses the junction. Current I_1 flows into the Gaussian Surface, and current I_2 flows out. Therefore, current density $\vec{J}_1$ points into the Gaussian Surface, and current density $\vec{J}_2$ points out. Since current density and electric field are related through the conductivity, $\vec{J} = \sigma\vec{E}$, field $\vec{E}_1$ points into the Gaussian Surface, and field $\vec{E}_2$ points out.

Since the area of the wire is uniform, the current densities must be the same ($J_1 = J_2 = J$), and the areas of flux in or out of the Gaussian Surface on either side of the junction must be the same ($A_1 = A_2 = A$). As flux is proportional to the charge inside the Gaussian Surface, a net positive (outward) flux is necessary for there to be a net positive charge inside. So

$$\Phi > 0 \quad \Rightarrow \quad E_2A - E_1A > 0 \quad \Rightarrow \quad E_2 > E_1 \quad \Rightarrow \quad \frac{J}{\sigma_2} > \frac{J}{\sigma_2} \quad \Rightarrow \quad \sigma_1 > \sigma_2$$