
I. (16 points) Consider the Bohr model of a hydrogen atom, an electron in a circular orbit about a proton. The radius of the orbit is 52.9×10^{-12} m. What minimum speed would the electron need to have for it to leave orbit and never return. If there is no such speed (that is, if the minimum speed is infinite), prove it.

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Use the Work-Energy Theorem:

$$W_{\text{ext}} = \Delta K + \Delta U + \Delta E_{\text{th}}$$

Choose a system that consists of the proton and electron. There are no external forces doing work on this system, so $W_{\text{ext}} = 0$. There are no non-conservative forces changing the thermal energy of this system, so $\Delta E_{\text{th}} = 0$.

$$0 = (K_f - K_i) + (U_f - U_i) + 0$$

The proton has much greater mass than the electron, so the kinetic energy change of the proton will be negligible. If the electron is never to return, it can't "stop" ($K_f = 0$) until it "gets to infinity" ($U_f = 0$).

$$0 = \left(0 - \tfrac{1}{2}mv_i^2\right) + \left(0 - \frac{Kq_1q_2}{r_i}\right)$$

where m is the mass of an electron, m_e. The proton and electron have charges e and $-e$, respectively, so $q_1q_2 = -e^2$.

$$\tfrac{1}{2}m_e v_i^2 = -\frac{K\left(-e^2\right)}{r_i}$$

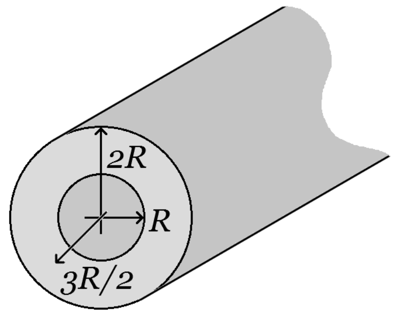
Note that the speed of the electron orbiting the proton is irrelevant. If the question had asked "How much work must be done to remove the electron from the atom?", then the kinetic energy of the orbiting electron would have to be considered. But since the question is just asking what speed is required for the electron to escape, the speed it has while orbiting doesn't matter. So

$$v_i = \sqrt{\frac{2Ke^2}{m_e r_i}} = \sqrt{\frac{2\left(8.988 \times 10^9\ \text{N}\cdot\text{m}^2/\text{C}^2\right)\left(1.602 \times 10^{-19}\ \text{C}\right)^2}{\left(9.109 \times 10^{-31}\ \text{kg}\right)\left(52.9 \times 10^{-12}\ \text{m}\right)}} = 3.09 \times 10^6\ \text{m/s}$$

II. (20 points) An infinite hollow insulating cylinder has inner radius R and outer radius $2R$, as illustrated. Its volume charge density, ρ, varies with distance r from the center according to

$$\rho = \rho_0 \left(\frac{R}{r}\right)^2$$

where ρ_0 is a positive constant. Find the electric field magnitude at a distance $3R/2$ from the center in terms of parameters defined in the problem, and physical or mathematical constants.



Use Gauss' Law, $\epsilon_0 \Phi_E = \epsilon_0 \oint \vec{E} \cdot d\vec{A} = q_{\text{in}}$. Choose a surface that passes through the point at which the electric field is to be found, and with the same symmetry as the charge distribution. A finite cylinder of radius $3R/2$ and length ℓ satisfies these conditions.

Note that the flux through the ends of the Gaussian Surface is zero, as the electric field vectors are perpendicular to the outward-pointing area vectors. The electric field has constant magnitude over the curved side of the Gaussian Surface, and is parallel to the outward-pointing area vectors.

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \oint E \cos\theta \, dA = E \cos(0^\circ) \oint dA = EA = E 2\pi r \ell = E 2\pi \left(\frac{3R}{2}\right) \ell = E 3\pi R \ell$$

The charge inside the Gaussian Surface can be found from the volume charge density, as $\rho = dq/dV$. Choose a thin cylindrical shell for the volume element, $dV = 2\pi r \ell \, dr$.

$$q_{\text{in}} = \int \rho \, dV = \int_R^{3R/2} \rho_0 \left(\frac{R}{r}\right)^2 2\pi r \ell \, dr = \rho_0 R^2 2\pi \ell \int_R^{3R/2} \frac{dr}{r} = \rho_0 R^2 2\pi \ell \ln \Big[r \Big]_R^{3R/2}$$

$$= \rho_0 R^2 2\pi \ell \Big[\ln(3R/2) - \ln(R) \Big] = \rho_0 R^2 2\pi \ell \ln\left(\frac{3R/2}{R}\right) = \rho_0 R^2 2\pi \ell \ln(3/2)$$

So

$$\epsilon_0 \Phi_E = q_{\text{in}} \qquad \Rightarrow \qquad \epsilon_0 E 3\pi R \ell = \rho_0 R^2 2\pi \ell \ln(3/2) \qquad \Rightarrow \qquad E = \frac{2\rho_0 R}{3\epsilon_0} \ln(3/2)$$

1. (6 points) In the problem above, what is the direction of the electric field at a distance $3R/2$ from the center?

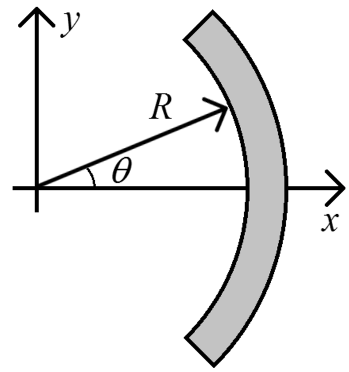
Since ρ_0 is positive, $\rho = \rho_0 (R/r)^2$ is also positive, and the hollow cylinder is positively charged. Electric field vectors point away from positive charge.

Away from the center.

2. (6 points) A non-uniform thin rod of charge is bent into an arc of radius R. It extends from $\theta = -\pi/4$ to $\theta = +\pi/4$, as shown. The linear charge density λ of the rod depends on angle θ according to

$$\lambda = \lambda_0 \theta^2$$

where λ_0 is a positive constant and θ is in radians. In what direction is the electric potential at the origin?



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Electric potential is a scalar!

This is not a meaningful question.

III. (16 points) In the problem above, what is the magnitude of the electric potential at the origin, with respect to zero at infinite distance? Express your answer in terms of parameters defined in the problem, and physical or mathematical constants.

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An element of charge dq makes an element of potential dV. Since the element of charge is point-like, the magnitude of the element of field is

$$dE = K\frac{dq}{r} \qquad \Rightarrow \qquad V = \int dV = \int K\frac{dq}{r}$$

Since the arc is circular, $r = R$, a constant. The element of charge can be related to θ through an element of arc length ds.

$$\lambda = \frac{dq}{ds} \qquad \Rightarrow \qquad dq = \lambda\, ds = \lambda_0 \theta^2\, R\, d\theta$$

So

$$V = \int_{-\pi/4}^{+\pi/4} K\frac{\lambda_0\theta^2\, R\, d\theta}{R} = K\lambda_0 \int_{-\pi/4}^{+\pi/4} \theta^2\, d\theta = K\lambda_0 \left[\frac{\theta^3}{3}\right]_{-\pi/4}^{+\pi/4}$$

$$= \frac{K\lambda_0}{3}\left[\left(\frac{\pi}{4}\right)^3 - \left(\frac{-\pi}{4}\right)^3\right] = \frac{K\lambda_0}{3}\left[\frac{2\pi^3}{64}\right] = \frac{K\lambda_0\pi^3}{96}$$

3. (8 points) The field outside a uniformly charged infinite solid cylinder of radius R is identical to that of an infinite line with equal linear charge density,

$$E = \frac{2K\lambda}{r}$$

when the distance from the central axis, r, is greater than or equal to R. If it can be determined, what is the electric field like *inside* the cylinder?

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Gauss' Law is always valid, even if it isn't always useful.

From symmetry, the field along the central axis must be zero. If it weren't, then rotating the cylinder about that axis would change the direction of the field on the axis, but would leave the charge distribution unchanged.

The field can't be zero everywhere inside the cylinder, as a Gaussian Surface within the cylinder contains some charge, so there is a net flux though it, and there must be non-zero field.

The only option consistent with these conditions is

The field increases linearly from zero at the center to $2K\lambda/R$ at the surface.

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4. (8 points) A conducting solid object has a bubble or void within it (**NOT** a hole drilled through it!). If the object has a charge Q, what charge q, if any, could be placed within the void to make the charges on the inner and outer surfaces of the object equal to each other?

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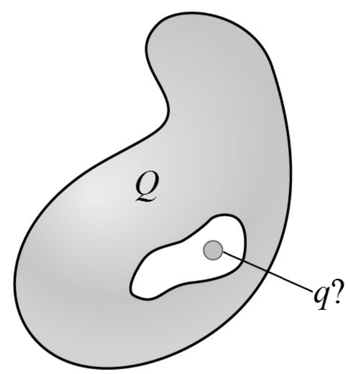
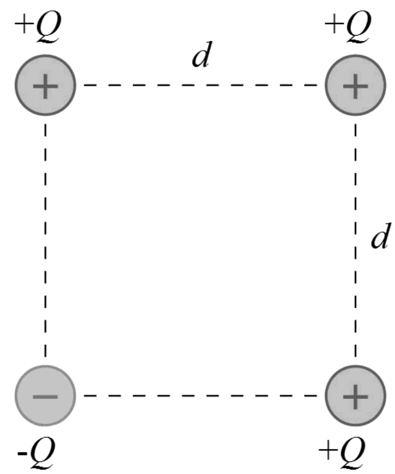


Since charge on the conducting solid object is conserved, for the charge on the inner and outer surfaces to be the same, each must have charge $Q/2$. But there's no electric field in the conducting solid at equilibrium, so a Gaussian Surface in the conducting solid and encompassing the void has no net flux through it, and so must contain no net charge. If there's charge $Q/2$ on the inner surface, but no net charge within the Gaussian Surface, the particle must have charge

$$q = -Q/2$$

5. (16 points) Four point particles are arranged on the corners a square, as shown. Each has the same charge magnitude Q, but three are positive while one is negative. If the square has sides of length d, what is the electric potential energy of the system of four charges, with respect to zero at infinite separation?



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The energy of the system is the sum of the energies of each pair of charged particles within it,

$$U = \frac{Kq_1q_2}{r}$$

The energy of the pair at the top of the square is equal and opposite that of the pair at the bottom of the square. The energy of the pair at the left of the square is equal and opposite that of the pair at the right of the square. The energy of the pair on the upper left to lower right diagonal of the square is equal and opposite that of the pair on the upper right to lower left diagonal of the square. Therefore, the total energy is

Zero

6. (8 points) A positively charged particle is placed against the positive plate of an ideal capacitor and released from rest. After it has traveled one-third the distance d across the capacitor, it has speed v_0. With what speed does it strike the negative plate?

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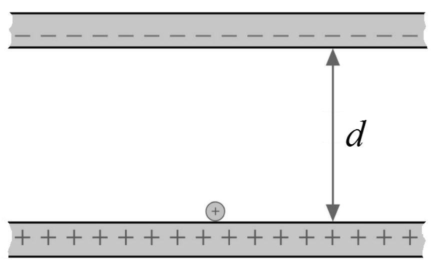

As the potential varies linearly with position in a parallel-plate capacitor, when the particle strikes the negative plate, it has passed through three times the potential difference as when it had only traveled one-third of the distance.

As the potential difference and potential energy difference are related by $\Delta U = q\,\Delta V$, when the particle strikes the negative plate, the potential energy of the particle-capacitor system has changed three times as much as when it had only traveled one-third of the distance.

As there are no external forces doing work on the particle-capacitor system, and no internal non-conservative forces changing the system's thermal energy, other energy in the system is conserved. When the particle strikes the negative plate, it must have three times the kinetic energy it did when it had traveled only one-third the distance.

$K = \frac{1}{2}mv^2$, so $v \propto \sqrt{K}$. The kinetic energy tripled, so the speed of the particle when it strikes the negative plate is

$$v_0\sqrt{3}$$

7. (8 points) A cube with sides of length L is positioned with one corner at the origin and its edges aligned with the xyz-axes as shown. There is a non-uniform electric field

$$\vec{E} = E_x\,\hat{\imath} - E_y\left(\frac{y}{L}\right)\hat{\jmath}$$

where E_x and E_y are positive constants. What is the net electric flux through the cube?

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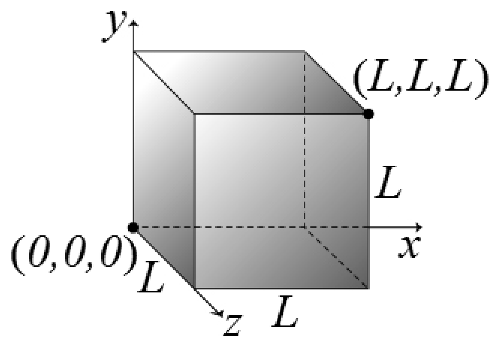

Electric flux is

$$\Phi_E = \int \vec{E}\cdot d\vec{A} = \int \vec{E}_X dA_X + \int \vec{E}_Y dA_Y + \int \vec{E}_Z dA_Z$$

This electric field has no z component, so there is no flux through the front and back faces of the cube. The x component of this electric field is constant, so the negative flux through the left face is equal and opposite the positive flux through the right face, and there's no net flux through those faces taken together.

The y component of this field is zero when $y = 0$, so there's no flux through the bottom face of the cube. The y component of the field is **not** zero when $y = L$, so there is non-zero flux through the top of the cube. Flux is a scalar, so the only option consistent with these conditions is

$$\Phi_E = -E_y L^2$$