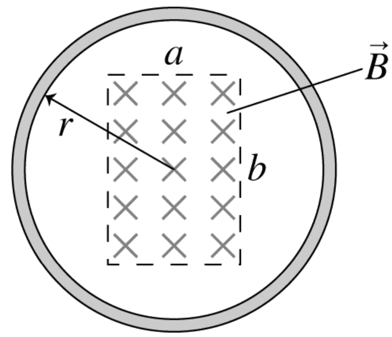
I. (16 points) A circular wire loop has radius r and resistance R. Within it, a rectangular region a wide and b high contains a magnetic field $\vec{B}$ directed into the page. The magnitude of this field depends on time t according to

$$B = B_0 \left(\frac{t}{t_0}\right)^2$$

where B_0 and t_0 are positive constants. What is the current in the loop at time $t = T$? Express your answer in terms of parameters defined in the problem, and physical or mathematical constants.



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From the definition of resistance, the current in the loop will be $I = \mathcal{E}/R$. The emf can be found using Faraday's Law.

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\left[\int \vec{B}\cdot d\vec{A}\right] = -\frac{d}{dt}\left[\int B\cos\theta\, dA\right]$$

As $\vec{B}$ and $d\vec{A}$ are parallel, $\cos\theta = 1$. Although the magnetic field depends on time, it does not depend on position, so, in terms of magnitudes

$$\mathcal{E} = \frac{d}{dt}\left[B\int dA\right] = \frac{d}{dt}\left[BA\right] = A\frac{dB}{dt} = ab\frac{d}{dt}\left[B_0\left(\frac{t}{t_0}\right)^2\right] = \frac{abB_0}{t_0^2}\frac{d}{dt}\left(t^2\right) = \frac{abB_0}{t_0^2}(2t)$$

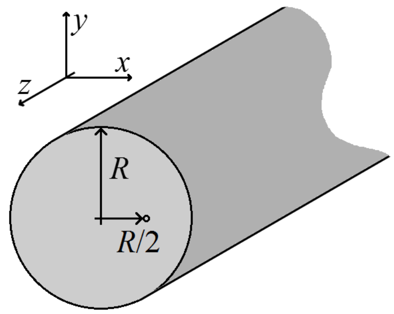
So, at time $t = T$, the current is

$$I = \frac{\mathcal{E}}{R} = \frac{2abB_0 t}{Rt_0^2} = \frac{2abB_0 T}{Rt_0^2}$$

II. (16 points) A cylindrical wire with radius R carries a current density $\vec{J}$ that depends on distance r from the cylinder axis according to

$$\vec{J} = J_0 \frac{r}{R} \hat{k}$$

where J_0 is a positive constant. What is the magnitude of the magnetic field at a distance $R/2$ from the center of the wire? Express your answer in terms of parameters defined in the problem, and physical or mathematical constants.



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Use Ampere's Law.

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{\text{thru}} = \mu_0 \int \vec{J} \cdot d\vec{A}$$

Choose an Amperian Loop that follows a magnetic field line and passes through the point at which the field is to be determined. This is a circle of radius $R/2$. The element of area for finding the current from the current density must be "small" in the direction of variation. Choose a thin ring of radius r and width dr.

$$B\,2\pi\left(\frac{R}{2}\right) = \mu_0 \int_0^{R/2} J_0 \frac{r}{R} \cos\theta \, 2\pi r \, dr = \frac{2\pi\mu_0 J_0}{R} \int_0^{R/2} r^2 \, dr = \frac{2\pi\mu_0 J_0}{R}\left[\frac{r^3}{3}\right]_0^{R/2} = \frac{2\pi\mu_0 J_0}{3R}\left(\frac{R}{2}\right)^3$$

where $\vec{J}$ and $d\vec{A}$ are parallel, so $\cos\theta = 1$. Solve for B.

$$B = \frac{\mu_0 J_0}{3R}\left(\frac{R}{2}\right)^2 = \frac{\mu_0 J_0 R}{12}$$

1. (6 points) In the problem above, what is the direction of the magnetic field at the indicated point right of the wire's center?

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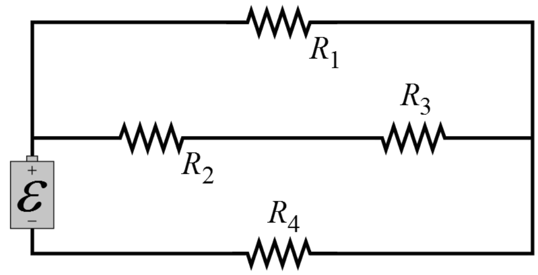
Using the short-cut Right-Hand-Rule, the thumb points in the direction of current flow (out of the page, the $+z$ direction), and the curled fingers point up the page at the indicated point, which is

In the $+y$ direction.

III. (16 points) The circuit shown is constructed with a 12 V battery, and four resistors having resistances

$R_1 = 32\,\Omega$
$R_2 = 14\,\Omega$
$R_3 = 18\,\Omega$
$R_4 = 8.0\,\Omega$



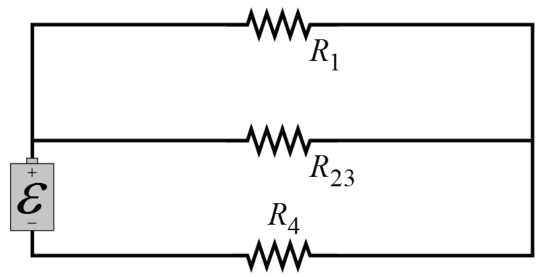
What is the power dissipated in resistor R_1?

The power dissipated by a resistor is

$$P = I\,\Delta V = I^2 R = (\Delta V)^2 / R$$

The potential across or current through R_1 can be determined by finding the equivalent resistance of the circuit. First, resistors R_2 and R_3 are in series (same current).

$$R_{23} = R_2 + R_3 = 14\,\Omega + 18\,\Omega = 32\,\Omega$$

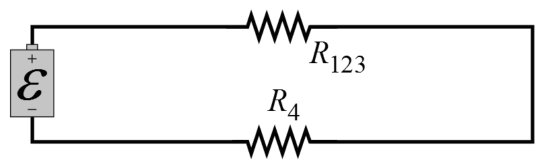


Next, R_1 and R_{23} are in parallel (same potential).

$$R_{123} = \left(\frac{1}{R_1} + \frac{1}{R_{23}}\right)^{-1} = \left(\frac{1}{32\,\Omega} + \frac{1}{32\,\Omega}\right)^{-1} = 16\,\Omega$$

Finally, R_{123} and R_4 are in series (same current).

$$R_{1234} = R_{123} + R_4 = 16\,\Omega + 8.0\,\Omega = 24\,\Omega$$



The current supplied by the emf is

$$I_{1234} = \frac{\Delta V_{1234}}{R_{1234}} = \frac{\mathcal{E}}{R_{1234}} = \frac{12\,\text{V}}{24\,\Omega} = 0.50\,\text{A}$$

As R_{123} and R_4 are in series, $I_{123} = I_4 = I_{1234} = 0.50\,\text{A}$.
The potential across R_{123} is

$$\Delta V_{123} = I_{123} R_{123} = (0.50\,\text{A})\,(16\,\Omega) = 8.0\,\text{V}$$

As R_1 and R_{23} are in parallel, $\Delta V_{123} = \Delta V_1 = \Delta V_{23} = 8.0\,\text{V}$.

So

$$P_1 = \frac{(\Delta V_1)^2}{R_1} = \frac{(8.0\,\text{V})^2}{32\,\Omega} = 2.0\,\text{W}$$

2. (6 points) In the problem above, let power dissipated in resistor R_1 be P_1. If a different battery were used in the circuit, with a potential difference twice that of the battery above (that is, $\mathcal{E}' = 2\mathcal{E}$), what would be the resulting power P_1' dissipated in resistor R_1?

The power dissipated in a resistor is $P = I\,\Delta V = (\Delta V)^2 / R$, so the power dissipated in R_1 is proportional to $(\Delta V_1)^2$, the square of the potential across R_1. From dimensional analysis, the potential across R_1 must be proportional to the emf. That is, ΔV_1 could be something like $2\mathcal{E}$ or $\mathcal{E}/\pi$, but it couldn't be something like $\mathcal{E}^3$ or $\ln \mathcal{E}$.

So, if the power dissipated in the resistor is proportional to the square of the potential across it, and the potential across the resistor is proportional to the emf, the power dissipated must change by the square of the factor change in the emf.

$$P_1' = 4P_1$$

3. (8 points) A circuit is constructed with a 48 V battery, a switch, three 16 Ω resistors, and a 30 mH inductor, as shown. The switch has been open for a long time. What current is supplied by the battery immediately upon closing the switch?

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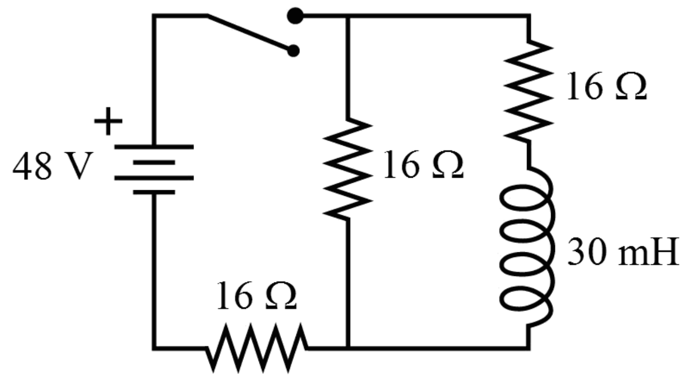

Inductors oppose changes in current. There is no current through the inductor before the switch is closed. Immediately after the switch is closed, then, there is still no current through the inductor. In that first instant, with no current flowing through the branch containing the inductor, the circuit is effectively a 48 V battery with two 16 Ω resistors in series (32 Ω total). The current is

$$I = \frac{\Delta V}{R} = \frac{48\,\mathrm{V}}{32\,\Omega} = 1.5\,\mathrm{A}$$

4. (8 points) A circuit is constructed with a 48 V battery, a switch, three 16 Ω resistors, and a 30 μF capacitor, as shown. The switch has been open for a long time. What current is supplied by the battery immediately upon closing the switch?

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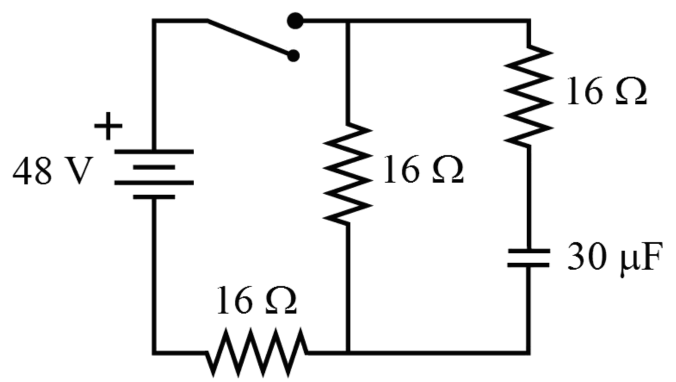

There is no charge on the capacitor before the switch is closed. Immediately after the switch is closed, there is still no charge on, and therefore no potential difference across, the capacitor. With no potential difference across it, the capacitor acts like a segment of ideal wire. In that first instant, the circuit is effectively a 48 V battery with two 16 Ω resistors in parallel (8 Ω equivalent) and that combination in series with a third 16 Ω resistor (24 Ω equivalent). The current is

$$I = \frac{\Delta V}{R} = \frac{48\,\mathrm{V}}{24\,\Omega} = 2.0\,\mathrm{A}$$

5. (8 points) Current flows counter-clockwise around a wire loop that is formed from two concentric circular arcs, connected by radii, as shown. What is the direction, if any, of the magnetic field at point P, which lies at the center of the two arcs?

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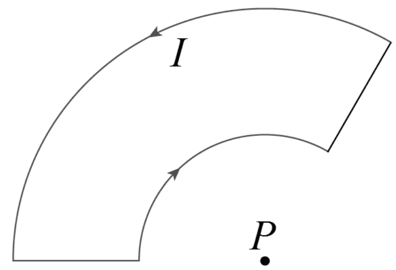

Use the Biot-Savart Law. The straight portions of the loop produce no field at the center, as the current is directly toward or away from the center. By the Right-Hand-Rule, the field due to the arc with smaller radius is in to the page, while the field due to the arc with greater radius is out of the page. To determine which has greater magnitude, find a general expression for the field at the center of a circular arc subtending an angle θ. In terms of magnitudes

$$B = \int dB = \int \frac{\mu_0 I}{4\pi} \frac{ds \sin\phi}{r^2} = \frac{\mu_0 I}{4\pi r^2} \int ds = \frac{\mu_0 I}{4\pi r^2} \int_0^\theta r\, d\theta = \frac{\mu_0 I}{4\pi r} \theta$$

where $\sin\phi = 1$ as $d\vec{s}$ is always perpendicular to a vector that points from $d\vec{s}$ to the center. The two arcs subtend the same angle θ. As the field due to each arc is inversely proportional to the radius, the field in to the page due to the smaller-radius arc has greater magnitude than the field out of the page due to the greater-radius arc. The net field, then, is

In to the page.

6. (8 points) A rectangular wire loop lies in the plane of the page with clockwise current I. A uniform magnetic field $\vec{B}$ is directed to the right, as shown. If there is a *net* torque on the loop, which side of the loop will tend to be lifted out of the page toward you?

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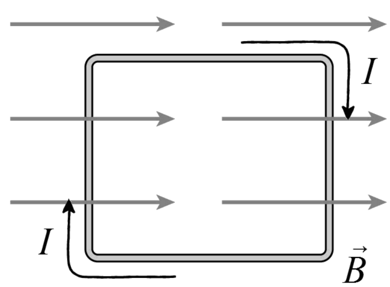

The magnetic moment of the loop points in the direction of the magnetic the current creates in the center of the loop. Using the short-cut Right-Hand-Rule, this is into the page. As the magnetic moment is not aligned with the external field directed to the right, there will be a net torque that tends to align the magnetic moment with the external field. So

A net torque will lift the *right* side out of the page.

7. (8 points) The plates in a capacitor have a potential difference of 2 kV between them. A particle of unstated charge is released from rest at the +2 kV plate and accelerates toward the 0 V plate. It passes through a small hole and enters a uniform magnetic field. If the particle's path is deflected up the page as shown, what is the direction of the magnetic field?

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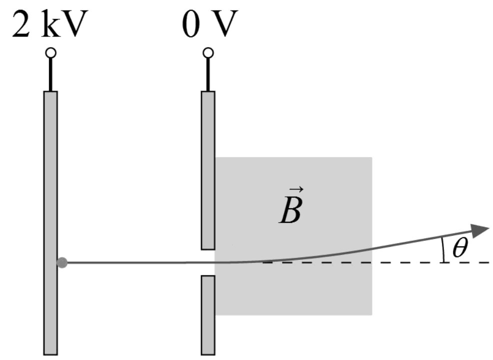

Electric field points from high potential to low potential, and from positive charge to negative charge. Therefore, the +2 kV plate is positively charged, and the 0 V plate is negatively charged. Since the particle accelerates from the +2 kV plate toward the 0 V plate, the particle must have positive charge.

The particle is moving to the right when it enters the magnetic field and experiences a magnetic force up the page. As the magnetic force on a moving charge is

$$\vec{F} = q\vec{v} \times \vec{B}$$

the Right-Hand-Rule shows us that the magnetic field must be

Into the page.