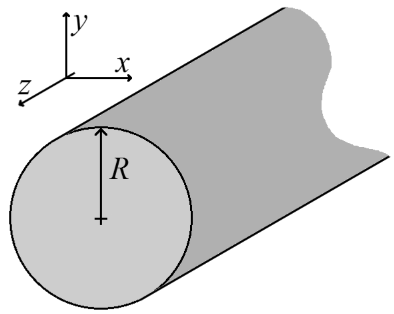

I. (16 points) A cylindrical wire with radius R carries a current density $\vec{J}$ that depends on distance r from the cylinder axis according to

$$\vec{J} = J_0 \frac{R}{r}\,\hat{k}$$

where $J_0 = 1.5 \times 10^5\ \mathrm{A/m^2}$. If the wire carries a total current of 6.0 A, what is the radius of the wire?



. . . .

Current is related to current density.

$$I = \int \vec{J} \cdot d\vec{A} = \int J \cos\theta \, dA$$

Choose an area element $d\vec{A}$ that is small in the direction of variation (r) and whose normal is parallel to the current density. This is a thin ring of radius r and width dr. It lies in the flat front face of the wire.

$$I = \int_0^R J_0 \frac{R}{r} (\cos 0)\, 2\pi r \, dr = 2\pi J_0 R \int_0^R dr = 2\pi J_0 R r \Big|_0^R = 2\pi J_0 R\,(R - 0) = 2\pi J_0 R^2$$

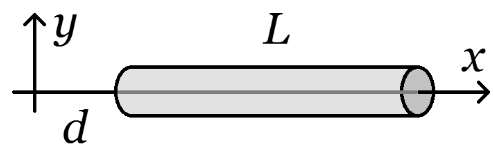
Solve for R.

$$R = \sqrt{\frac{I}{2\pi J_0}} = \sqrt{\frac{6.0\ \mathrm{A}}{2\pi\,(1.5 \times 10^5\ \mathrm{A/m^2})}} = 2.5 \times 10^{-3}\ \mathrm{m}$$

II. (16 points) A thin rod with length L lines on the $+x$ axis, with one end at $x = d$ and the other at $d + L$, as shown. Its linear charge density λ depends of position x according to

$$\lambda = \lambda_0 \left(\frac{d}{x}\right)$$

where λ_0 is a positive constant. What is the magnitude of the electric potential at the origin, with respect to zero at infinite distance? Express your answer in terms of parameters defined in the problem, and physical or mathematical constants.



. . . .

Choose point-like elements of charge dq, each of which contributes an element of potential, dV, at the origin. A thin slice of the rod, with width dx, has charge

$$dq = \lambda\, dx \qquad \text{since} \qquad \lambda = \frac{dq}{dx}$$

So, as the elements of charge are point-like

$$V = \int dV = \int K \frac{dq}{r} = \int K \frac{\lambda\, dx}{x} = \int_d^{d+L} K \frac{\lambda_0 \,(d/x)\; dx}{x} = K\lambda_0 d \int_d^{d+L} x^{-2}\, dx = K\lambda_0 d \int_d^{d+L} x^{-2}\, dx$$

$$= K\lambda_0 d \left[\frac{x^{-1}}{-1}\right]_d^{d+L} = -K\lambda_0 d \left[\frac{1}{d+L} - \frac{1}{d}\right] = K\lambda_0 d \left[\frac{d+L}{d\,(d+L)} - \frac{d}{d\,(d+L)}\right] = K\lambda_0 d \left[\frac{L}{d\,(d+L)}\right]$$

$$= \frac{K\lambda_0 L}{d+L}$$

1. (6 points) In the problem above, what is the direction, if any, of the electric potential at the origin?

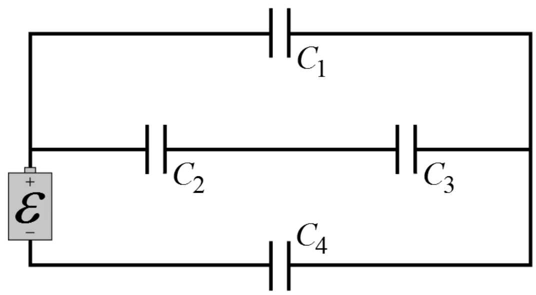
. . . .

Electric potential is a scalar!

No direction, as this is not a meaningful question.

III. (16 points) Each of the four capacitors in the circuit shown has a identical capacitance C. The battery has a potential difference $\mathcal{E}$. What is the potential difference across capacitor C_1? Express your answer in terms of parameters defined in the problem, and physical or mathematical constants.

. . . .

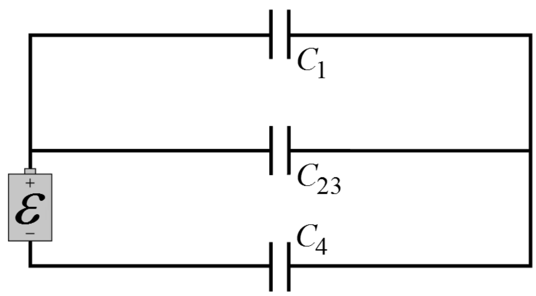


Find the equivalent capacitance by combining capacitors in series or parallel. Then work back toward the original circuit, finding charges and potentials. First, combine capacitors C_1 and C_2 in series (same charge).

$$C_{23} = \left(\frac{1}{C_2} + \frac{1}{C_3}\right)^{-1} = \left(\frac{1}{C} + \frac{1}{C}\right)^{-1} = \left(\frac{2}{C}\right)^{-1} = C/2$$

Next, combine capacitors C_1 and C_{23} in parallel (same potential).

$$C_{123} = C_1 + C_{23} = C + C/2 = 3C/2$$



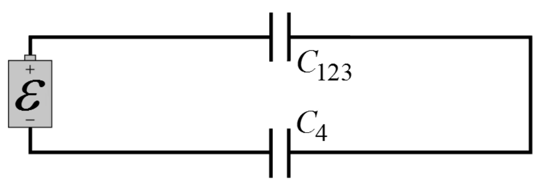
Finally, combine capacitors C_{123} and C_4 in series.

$$C_{1234} = \left(\frac{1}{CC_{123}} + \frac{1}{C_4}\right)^{-1} = \left(\frac{1}{3C/2} + \frac{1}{C}\right)^{-1}$$

$$= \left(\frac{2}{3C} + \frac{3}{3C}\right)^{-1} = \left(\frac{5}{3C}\right)^{-1} = 3C/5$$

The potential across this equivalent capacitance is the emf of the battery, so the charge on C_{1234} is

$$Q_{1234} = C_{1234}\,\Delta V_{1234} = \frac{3C}{5}\,\mathcal{E} = 3C\mathcal{E}/5$$



This must also be the charge on C_{123}, as C_{123} and C_4 are in series. The potential across C_{123}, then, is

$$\Delta V_{123} = \frac{Q_{123}}{C_{123}} = \frac{Q_{1234}}{C_{123}} = \frac{3C\mathcal{E}/5}{3C/2} = 2\mathcal{E}/5$$

This must also be the potential across C_1, as C_1 and C_{23} are in parallel, so ... $\Delta V_1 = \Delta V_{123} = 2\mathcal{E}/5$

2. (6 points) In the problem above, let the charge on capacitor C_1 be Q_1. If a different battery were used in the circuit, with a potential difference twice that of the battery above (that is, $\mathcal{E}' = 2\mathcal{E}$), what would be the resulting charge Q_1' on capacitor C_1?

. . . .

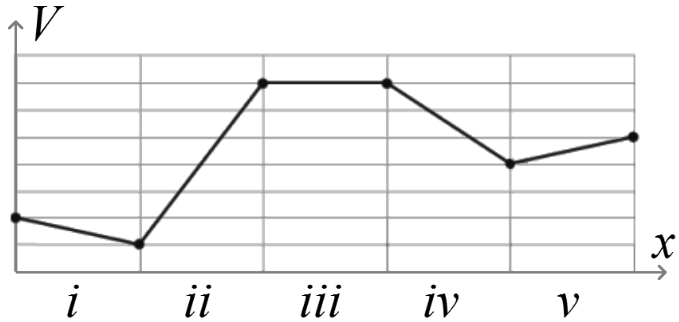
From the definition of capacitance, $Q = C\,\Delta V$, the charge on capacitor C_1 is proportional to ΔV_1, the potential across C_1. From dimensional analysis, the potential across C_1 must be proportional to the emf. That is, ΔV_1 could be something like $2\mathcal{E}$ or $\mathcal{E}/\pi$, but it couldn't be something like $\mathcal{E}^3$ or $\ln \mathcal{E}$.

So, if the charge on the capacitor is proportional to the potential across it, and the potential across the capacitor is proportional to the emf, the change must change by the same factor as the emf.

$$Q_1' = 2Q_1$$

3. (8 points) Electric potential, V, in a region of space depends on position, x, as shown. In which of the five indicated regions does the electric field have its maximum magnitude?

. . . .



Potential is related to field by

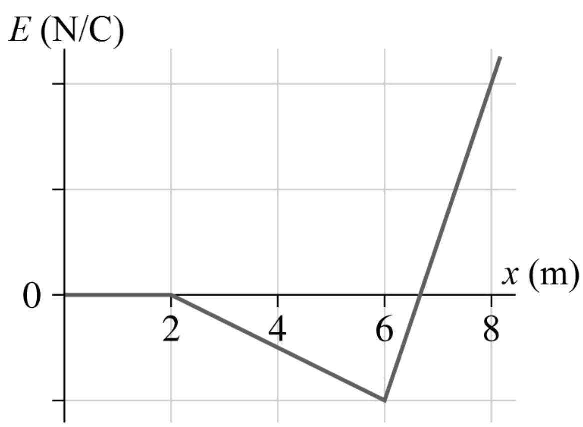
$$E_s = -\frac{\delta V}{\delta s}$$

so the field is the slope of the graph of potential as a function of position. The maximum field magnitude will be in the region of steepest slope, regardless of sign, which is

In region *ii*.

4. (8 points) The electric field in a region of space depends on position x, as shown. If the electric potential is zero at $x = 0$, at what location in the range $x = 0$ to $x = 8.0\,\text{m}$ does the electric potential have its maximum value?

. . . .



Potential is related to field by

$$\Delta V = -\int \vec{E} \cdot d\vec{s}$$

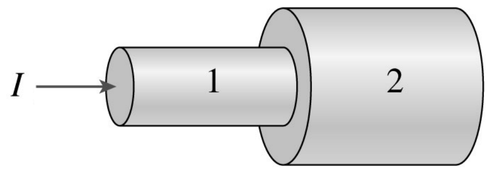
so the potential difference is the opposite of the area under the graph of field as a function of position. The maximum potential difference from the origin will be at the point where the most *negative* area under the curve has been accumulated, which is

At $x = 6.7\,\text{m}$.

---

5. (8 points) The wire shown consists of two segments of different diameters made from different materials. The conductivity of segment 2 is half that of segment 1 (that is, $\sigma_2 = \sigma_1/2$). If segment 1 has radius R_1, what radius of segment 2 would result in each segment having the same electric field?

. . . .



The conductivity relates the electric field and the current density in a wire.

$$\vec{J} = \sigma \vec{E} \qquad \Rightarrow \qquad E = \frac{J_1}{\sigma_1} = \frac{J_2}{\sigma_2} = \frac{I_1/A_1}{\sigma_1} = \frac{I_2/A_2}{\sigma_2}$$

Applying Kirchhoff's Junction Law to the connection between the segments shows that the current must be the same in each segment ($I_1 = I_2$).

$$\frac{I}{\pi R_1^2 \sigma_1} = \frac{I}{\pi R_2^2 \sigma_2} \qquad \Rightarrow \qquad R_2 = R_1\sqrt{\frac{\sigma_1}{\sigma_2}} = R_1\sqrt{\frac{\sigma_1}{\sigma_1/2}} \qquad \Rightarrow \qquad R_2 = R_1\sqrt{2}$$

---

6. (8 points) A parallel-plate capacitor is connected to a battery with emf $\mathcal{E}$. While it remains connected, insulating handles are used to push the plates closer together. As the distance between the plates decreases, how is the charge magnitude on the plates affected? How is the electric field magnitude between the plates affected?

. . . .

As the capacitor remains connected to the battery, the potential across the capacitor cannot change. From the capacitance of a parallel-plate capacitor and the definition of capacitance

$$C = \epsilon_0 \frac{A}{d} \qquad \text{and} \qquad Q = C\,\Delta V \qquad \Rightarrow \qquad Q = \epsilon_0 \frac{A}{d}\Delta V$$

if the distance d between the plates is reduced, the charge magnitude on the plates must increase. The electric field magnitude in a parallel-plate capacitor is

$$E = \frac{\Delta V}{d} = \frac{\eta}{\epsilon_0} = \frac{Q/A}{\epsilon_0}$$

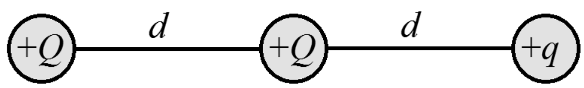
so if the distance d between the plates is reduced (or, equivalently in this case, the charge on the capacitor is increased) the electric field magnitude must increase.

Charge magnitude increases. Electric field magnitude increases.

7. (8 points) Three positively-charged particles, two with charge $+Q$ and one with charge $+q$ are arranged as shown. Each has mass m. The particles with charge $+Q$ are held in place, while the particle with charge $+q$ is released from rest. What is the speed of the particle with charge $+q$ when it is very far from the other particles?

. . . .

Use the Work-Energy Theorem, $W_{\text{ext}} = \Delta K + \Delta U + \Delta E_{\text{th}}$.



Choose a system consisting of the three charged particles. No external forces do work on this system, and no internal non-conservative forces change the thermal energy.

$$0 = (K_f - K_i) + (U_f - U_i) + 0 \qquad \Rightarrow \qquad 0 = (K_f - K_i) + q\,(V_f - V_i)$$

None of the particles are moving initially, so $K_i = 0$. Only the particle with charge $+q$ is moving finally, so $K_f = \frac{1}{2}mv_f^2$. Let the potential be zero at infinite distance, so $V_f = 0$ and KQ/r can be used to calculate the potential at the initial location of the $+q$ particle.

$$0 = (K_f - 0) + q\,(0 - V_i) \qquad \Rightarrow \qquad \tfrac{1}{2}mv_f^2 = qV_i$$

The potential at the initial location of the $+q$ particle is the sum of the potentials due to the two $+Q$ particles.

$$V_i = \frac{KQ}{2d} + \frac{KQ}{d} = 3\frac{KQ}{2d}$$

So

$$\tfrac{1}{2}mv_f^2 = q\left(3\frac{KQ}{2d}\right) \qquad \Rightarrow \qquad v_f = \sqrt{3\frac{K}{m}\frac{Qq}{d}}$$