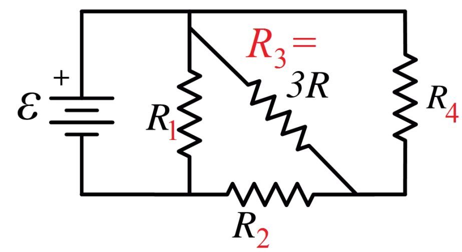
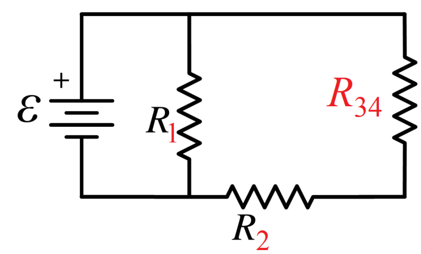
I. (16 points) The circuit shown has an emf $\mathcal{E}$, three resistors with resistance R, and one resistor with resistance $3R$. What is the current through the resistor with resistance $3R$? Express your answer in terms of parameters defined in the problem, and physical or mathematical constants.



Resistors have been numbered for clarity. Combine them in series or parallel, as appropriate. First, resistors R_3 and R_4 are in parallel (same potential difference).

$$R_{34} = \left(\frac{1}{R_3} + \frac{1}{R_4}\right)^{-1} = \left(\frac{1}{3R} + \frac{1}{R}\right)^{-1}$$

$$= \left(\frac{1}{3R} + \frac{3}{3R}\right)^{-1} = \left(\frac{4}{3R}\right)^{-1} = \frac{3R}{4}$$

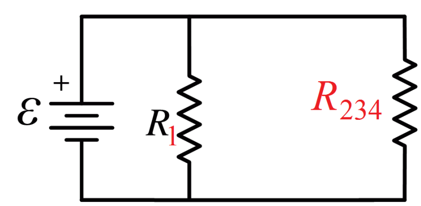


Next, resistors R_2 and R_{34} are in series (same current).

$$R_{234} = R_2 + R_{34} = R + \frac{3R}{4} = \frac{7R}{4}$$

The potential difference across R_{234} is $\mathcal{E}$, so the current through it is

$$I_{234} = \frac{\Delta V_{234}}{R_{234}} = \frac{\mathcal{E}}{7R/4} = \frac{4\mathcal{E}}{7R}$$



That must also be the current through R_{34}, so the potential across it is

$$\Delta V_{34} = I_{34}R_{34} = I_{234}R_{34} = \left(\frac{4\mathcal{E}}{7R}\right)\left(\frac{3R}{4}\right) = \frac{3\mathcal{E}}{7}$$

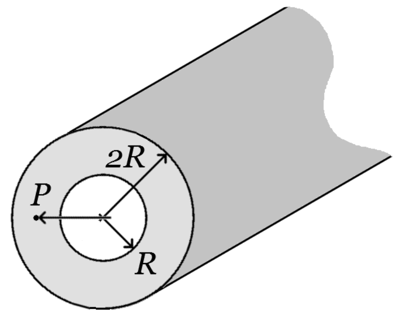
That must also be the potential across R_3, so the current through it is

$$I_3 = \frac{\Delta V_3}{R_3} = \frac{\Delta V_{34}}{R_3} = \frac{3\mathcal{E}/7}{3R} = \frac{\mathcal{E}}{7R}$$

II. (16 points) An infinite straight hollow wire has inner radius R and outer radius $2R$, as illustrated. Its current density, $\vec{J}$, is directed into the page and has a magnitude that varies with distance r from the center according to

$$J = J_0 \frac{R^2}{r^2}$$

where J_0 is a positive constant. Find the magnetic field magnitude at a point P which is a distance $3R/2$ from the center, in terms of parameters defined in the problem, and physical or mathematical constants.



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Use Ampere's Law, choosing a circular path passing through the point P and centered on the cylinder axis. This choice gives $\vec{B}$ a constant magnitude, and a direction that is everywhere parallel to the path element $\vec{ds}$.

$$\oint \vec{B} \cdot \vec{ds} = \mu_0 I_{\text{thru}} \qquad \Rightarrow \qquad B \oint ds = \mu_0 \int \vec{J} \cdot \vec{dA}$$

Let the area element $\vec{dA}$ be a thin ring with a direction parallel to the current density $\vec{J}$. Note that $\oint ds$ is the circumference of the circle constituting the path.

$$B 2\pi \left(\frac{3R}{2}\right) = \mu_0 \int_R^{3R/2} J_0 \frac{R^2}{r^2} 2\pi r \, dr = \mu_0 J_0 2\pi R^2 \int_R^{3R/2} \frac{dr}{r} = \mu_0 J_0 2\pi R^2 \ln r \Big|_R^{3R/2} = \mu_0 J_0 2\pi R^2 \ln(3/2)$$

So

$$B = \frac{2\mu_0 J_0 R}{3} \ln(3/2)$$

1. (6 points) In the problem above, what is the direction of the magnetic field at the illustrated point P?

. . . .

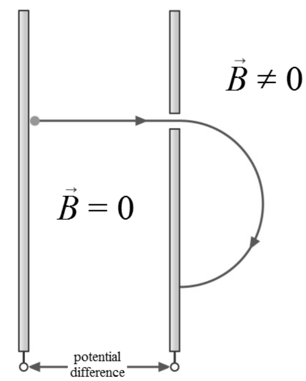
Using either the Biot-Savart Law for current

$$\vec{dB} = \frac{\mu_0 I}{4\pi} \frac{\vec{d\ell} \times \hat{r}}{|\vec{r}|^2}$$

with $\vec{d\ell}$ into the page and $\hat{r}$ to the left, or the equivalent Right-Hand-Rule shortcut with thumb pointed into the page and curled fingers indicating the magnetic field direction, one finds that the field at P is

Up the page.

2. (6 points) A positively-charged particle is accelerated from rest through a potential difference, then passes through a region of uniform magnetic field with magnitude B. It follows a half-circle, as illustrated, and strikes the plate defining the edge of the field.



What is the direction of the magnetic field?

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The force on a particle moving through a magnetic field is $\vec{F} = q\vec{v} \times \vec{B}$. When this positively-charged particle first enters the field, it is moving to the right and the force is down the page. The Right-Hand-Rule for the vector product shows us that the field must be directed

Out of the page.

III. (16 points) The particle in the problem above has mass m, charge magnitude q, and enters the magnetic field with speed v. At what time Δt after entering the field does the particle strike the plate? Express your answer in terms of parameters defined in the problem, and physical or mathematical constants.

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While the particle is in the magnetic field, the only force on it is the magnetic force. The path of the particle is a segment of a circle, so the magnetic force is a centripetal force. Use Newton's Second Law.

$$\sum F_c = F_B = ma_c \qquad \Rightarrow \qquad qvB\sin\phi = m\frac{v^2}{r}$$

The velocity is always perpendicular to the field, so $\phi = 90°$ and $\sin\phi = 1$. The magnitude of the velocity is the path through which the particle has moved, divided by the time required, $v = s/\Delta t$.

$$qvB\sin\phi = m\frac{v^2}{r} \qquad \Rightarrow \qquad qB = m\frac{s/\Delta t}{r}$$

The particle strikes the plate after moving through a half circle, so $s = \pi r$.

$$qB = m\frac{\pi r}{r\,\Delta t} \qquad \Rightarrow \qquad \Delta t = \frac{\pi m}{qB}$$

3. (8 points) A current I flows counterclockwise around a loop constructed of two quarter-circle arcs joined as shown. The inner arc has radius R_i and the outer arc has radius R_o. What is the direction of the magnetic field at point P, the center of the arcs?

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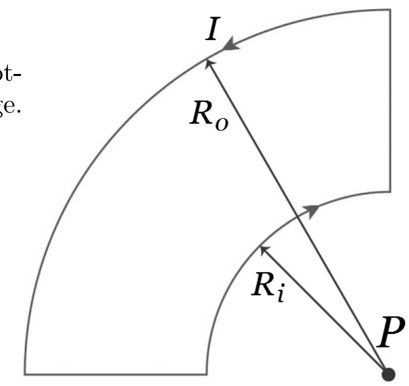

(Expand on $B_i > B_o$.)Applying the Right-Hand-Rule to $\vec{d\ell} \times \hat{r}$ in the Biot-Savart Law, we find that B_i is into the page and B_o is out of the page. Since $B_i > B_o$, the net field is

Into the page.

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4. (8 points) The illustrated slab is metal, so the charge carriers are electrons. It lies in a uniform magnetic field and *conventional current* flows through it to the right (in the $+y$ direction). The visible front face is at lower electric potential than the hidden back face. In what direction does the magnetic field point?

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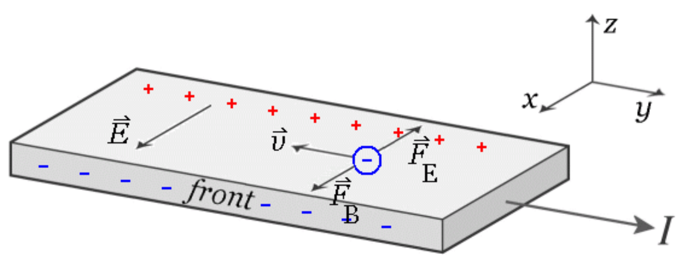


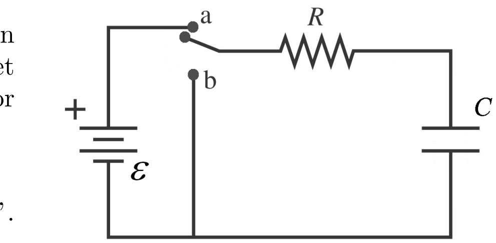
Because we know the front is at lower electric potential, the electric field $\vec{E}$ must point from back to front. Because electric field points from positive charge to negative charge, there must be negative charge at the front of the conductor, and positive charge at the back. The negative charge at the front of the conductor got there because the magnetic force $\vec{F}_B$ pushed electrons (which are what moves in a metal conductor, with a velocity $\vec{v}$ opposite the direction of conventional current) there. Since the directions of $\vec{v}$ and $\vec{F}_B$ are known, the direction of the magnetic field can be found using $\vec{F}_B = q\vec{v} \times \vec{B}$. Because q is negative, $\vec{F}_B$ is in the opposite direction from $\vec{v} \times \vec{B}$. Therefore, $\vec{B}$ must be in the direction

$+z$

5. (8 points) The switch in the illustrated circuit is set to position "b" for a long time, then set to position "a" for a time t_a, then set back to position "b". After that, the current through the resistor is

$$I = I_0 e^{-t_b/RC}$$

where t_b is the time from returning the switch to position "b". What is I_0?



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Charge on the capacitor remains constant as the switch is thrown from "a" to "b". At the end of time t_a, that charge is

$$Q(t_a) = Q_\infty \left(1 - e^{t_a/RC}\right) = C\mathcal{E}\left(1 - e^{t_a/RC}\right)$$

So

$$\Delta V_C(t_a) = Q(t_a)/C = \mathcal{E}\left(1 - e^{t_a/RC}\right)$$

Once the switch is thrown back to "b", the potential difference across the resistor must be the same as the potential difference across the capacitor.

$$I = \Delta V_R/R \quad \Rightarrow \qquad I_0 = \Delta V_C(t_a)/R = (\mathcal{E}/R)\left(1 - e^{-t_a/RC}\right)$$

6. (8 points) Resistors R_1 through R_5 in the illustrated circuit carry currents I_1 through I_5, respectively. Let the positive direction of current flow be left to right through all the resistors. Which equation is a valid expression of Kirchhoff's Loop Law?

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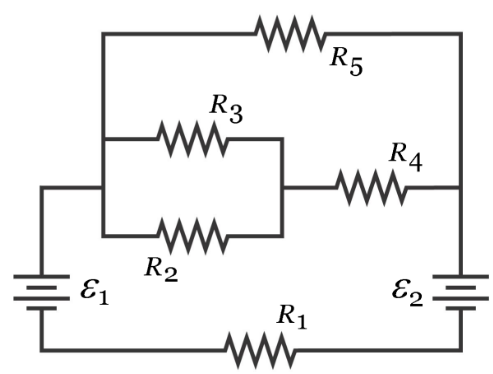
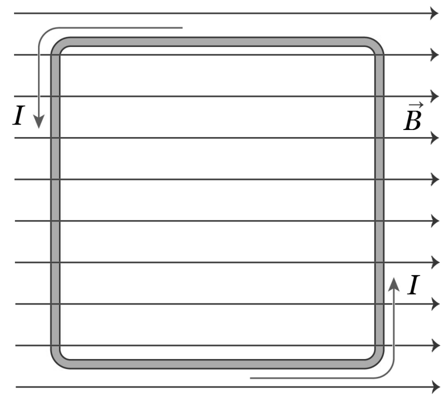


With the positive direction of current flow defined as left to right, traversing a resistor from left to right should be represented as a *decrease* in potential, $-IR$. Check each possible answer starting with $+\mathcal{E}_1$. If an incorrect potential change is encountered, or the answer does not represent a closed loop, that answer must be rejected. The only option presented which represents a closed loop and has all potential changes correct is

$$+\mathcal{E}_1 - I_3R_3 - I_4R_4 - \mathcal{E}_2 + I_1R_1 = 0$$

7. (8 points) A square wire loop lies in the plane of the page, carrying a counterclockwise current. A uniform magnetic field is directed to the right, as illustrated. What effect does the net torque, if any, have on the loop?



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The force on a current-carrying wire in a magnetic field is

$$d\vec{F} = I\, d\vec{\ell} \times \vec{B}$$

The forces on the top and bottom sides of the loop are zero, as the current is parallel or anti-parallel with the field. If each side of the loop has length ℓ, then the force on the left side of the loop is $I\ell B$ out of the page, and the force on the right side of the loop is $I\ell B$ in to the page. Therefore

The net torque lifts the left side of the loop out of the page.