
I. (16 points) An electric field varies with position according to

$$\vec{E} = \left(6.0\,\mathrm{V/m^2}\right) x\,\hat{\imath} - \left(3.0\,\mathrm{V/m^3}\right) y^2\,\hat{\jmath}$$

What is the electric potential difference **from** the point $(x_1, y_1) = (1.0, 2.0)\,\mathrm{m}$ **to** the point $(x_2, y_2) = (2.0, -2.0)\,\mathrm{m}$?

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The electric potential difference can be related to the electric field. Let $A = 6.0\,\mathrm{V/m^2}$ and $B = 3.0\,\mathrm{V/m^3}$.

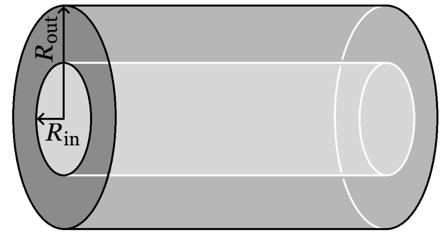
$$\Delta V = V_f - V_i = -\int \vec{E}\cdot\vec{ds} = -\int E_x\,dx - \int E_y\,dy$$

$$= -\int_{x_1}^{x_2} Ax\,dx - \int_{y_1}^{y_2}\left(-By^2\right)dy = -\left.\frac{Ax^2}{2}\right|_{x_1}^{x_2} + \left.\frac{By^3}{3}\right|_{y_1}^{y_2} = \frac{-A}{2}\left[x_2^2 - x_1^2\right] + \frac{B}{3}\left[y_2^3 - y_1^3\right]$$

$$= \frac{-6.0\,\mathrm{V/m^2}}{2}\left[(2.0\,\mathrm{m})^2 - (1.0\,\mathrm{m})^2\right] + \frac{3.0\,\mathrm{V/m^3}}{3}\left[(-2.0\,\mathrm{m})^3 - (2.0\,\mathrm{m})^3\right] = -9.0\,\mathrm{V} - 16\,\mathrm{V}$$

$$= -25\,\mathrm{V}$$

II. (16 points) A hollow conducting wire has inner radius R_{in} and outer radius R_{out}. It carries a current whose density magnitude J varies with distance r from the central axis of the wire according to



$$J = J_0 \frac{r - R_{\text{in}}}{r}$$

where J_0 is a positive constant. In terms of parameters defined in the problem and physical or mathematical constants, what is the magnitude of the current in the wire?

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The relationship between current and current density is

$$I = \int \vec{J} \cdot d\vec{A}$$

Since $\vec{J}$ varies with r, choose an area element that is small in the "r" direction. This will be a thin ring of area magnitude $dA = 2\pi r\, dr$. Note that $\vec{J}$ is parallel to this $d\vec{A}$, so

$$I = \int \vec{J} \cdot d\vec{A} = \int J\, dA = \int_{R_{\text{in}}}^{R_{\text{out}}} J_0 \frac{r - R_{\text{in}}}{r}\, 2\pi r\, dr = 2\pi J_0 \int_{R_{\text{in}}}^{R_{\text{out}}} (r - R_{\text{in}})\, dr = 2\pi J_0 \left[\frac{r^2}{2} - R_{\text{in}} r\right]_{R_{\text{in}}}^{R_{\text{out}}}$$

$$= \pi J_0 \left[r^2 - 2R_{\text{in}} r\right]_{R_{\text{in}}}^{R_{\text{out}}} = \pi J_0 \left[\left(R_{\text{out}}^2 - 2R_{\text{in}} R_{\text{out}}\right) - \left(R_{\text{in}}^2 - 2R_{\text{in}} R_{\text{in}}\right)\right] = \pi J_0 \left[R_{\text{out}}^2 - 2R_{\text{in}} R_{\text{out}} + R_{\text{in}}^2\right]$$

$$= \pi J_0 \left[R_{\text{out}} - R_{\text{in}}\right]^2$$

1. (6 points) In the problem above, at what distance, if any, from central axis is the drift speed of electrons in the wire zero?

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Since

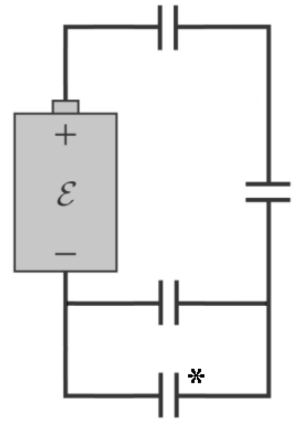
$$v_d \propto E \propto J$$

this question is asking at what distance from the central axis is the current density zero.

$$J = J_0 \frac{r - R_{\text{in}}}{r} = 0 \qquad \Rightarrow \qquad r = R_{\text{in}}$$

(only).

III. (16 points) The battery in the illustrated circuit has emf $\mathcal{E}$. Each of the four capacitors has the same capacitance, C. Once the circuit has been connected for a long time, what energy is stored in the bottom-most capacitor, marked with an asterisk, with respect to zero energy at zero charge? Express your answer in terms of parameters defined in the problem, and physical or mathematical constants.



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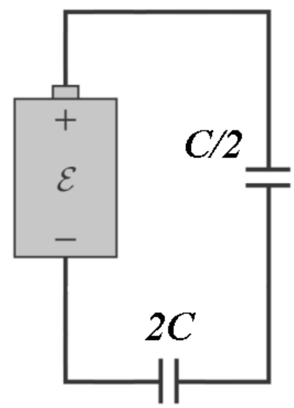
The top and right-most capacitors are in series (same charge). Their equivalent capacitance can be found.

$$\frac{1}{C_{\text{eq1}}} = \frac{1}{C} + \frac{1}{C} \qquad \Rightarrow \qquad C_{\text{eq1}} = \frac{C}{2}$$

The two bottom capacitors are in parallel (same potential difference). Their equivalent capacitance can be found.

$$C_{\text{eq2}} = C + C = 2C$$



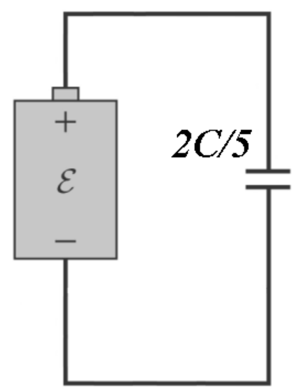
These two new equivalent capacitances are in series. The equivalent capacitance for the entire circuit can be found.

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C/2} + \frac{1}{2C} \qquad \Rightarrow \qquad C_{\text{eq}} = \frac{2C}{5}$$

The charge delivered by the battery is

$$Q = C_{\text{eq}}\,\Delta V = \left(\frac{2C}{5}\right)\mathcal{E}$$

This charge is on the $2C/5$ equivalent capacitor, and on each of the $C/2$ and $2C$ equivalent capacitors (since they are in series). The potential difference across the $2C$ equivalent capacitor is



$$\Delta V = \frac{Q}{C_{\text{eq2}}} = \frac{(2C/5)\,\mathcal{E}}{2C} = \frac{\mathcal{E}}{5}$$

Since the two bottom capacitors are in parallel, this is also the potential difference across each of them individually. The energy stored in the bottom-most capacitor, then, is

$$U = \tfrac{1}{2}C\,(\Delta V)^2 = \tfrac{1}{2}C\left(\frac{\mathcal{E}}{5}\right)^2 = \frac{C\mathcal{E}^2}{50}$$

2. (6 points) If the energy found in the problem above is U_0, what energy would be stored in that same capacitor if the battery were replaced by one with twice the emf, $2\mathcal{E}$?

When the battery has emf $2\mathcal{E}$, the energy stored in the bottom-most capacitor is . . .

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The stored energy is proportional to the square of the potential across the capacitor. In a circuit with only one battery, the energy stored in each capacitor must be proportional to the square of the battery's emf. The energy stored in the bottom-most capacitor is

$$4U_0$$

3. (8 points) The electric potential has been graphed as a function of position along the x axis. At which of the indicated points does the electric field point in the negative x direction with the greatest magnitude?

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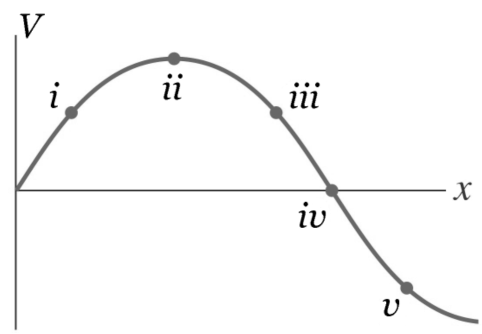

The electric field can be related to the electric potential by

$$E = -\frac{\delta V}{\delta s}$$

Graphically, this means that the electric field is the opposite of the slope of the graph of electric potential as a function of position. The electric field points in the negative direction when the slope of the graph is positive. The only one of the indicated points at a position of positive slope is

Point i

4. (8 points) Each of the circuits shown has an identical emf $\mathcal{E}$. Rank the circuits in order of power delivered to resistor R_1.

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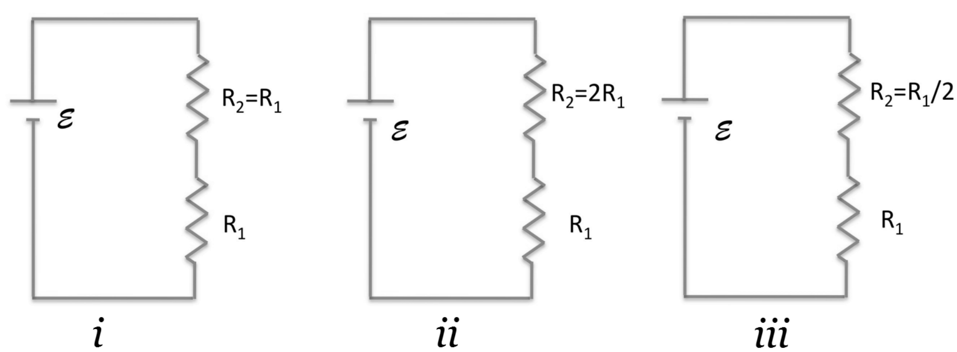

Power dissipated in a resistor is

$$P = I\,\Delta V = I^2 R$$

so the greatest power will be dissipated in the R_1 that has the greatest current.

All the circuits have the same emf $\mathcal{E}$, so the greatest current will flow in the circuit with the least equivalent resistance. All the circuits have two resistances in series, so

$$R_{\mathrm{eq}\,i} = R_1 + R_1 = 2R_1 \qquad R_{\mathrm{eq}\,ii} = 2R_1 + R_1 = 3R_1 \qquad R_{\mathrm{eq}\,iii} = \frac{R_1}{2} + R_1 = \frac{3}{2}R_1$$

So from greatest to least, the powers delivered to R_1 in the three circuits are

$$iii > i > ii$$

5. (8 points) A tungsten wire and an aluminum wire are joined to make a single wire. A current I enters the aluminum wire. The radius of the tungsten wire is twice that of the aluminum end. The conductivity of tungsten is half that of aluminum. How is the electric field magnitude in the tungsten wire E_{W} related to that in the aluminum wire E_{Al}?

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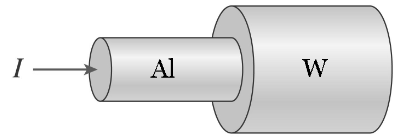
The current in each portion of the wire is the same.



$$I_{\mathrm{Al}} = I_{\mathrm{W}} \quad \Rightarrow \quad J_{\mathrm{Al}} A_{\mathrm{Al}} = J_{\mathrm{W}} A_{\mathrm{W}} \quad \Rightarrow \quad \sigma_{\mathrm{Al}} E_{\mathrm{Al}} \pi r_{\mathrm{Al}}^2 = \sigma_{\mathrm{W}} E_{\mathrm{W}} \pi r_{\mathrm{W}}^2$$

Solving for the field magnitude in tungsten,

$$E_{\mathrm{W}} = E_{\mathrm{Al}} \frac{\sigma_{\mathrm{Al}} \pi r_{\mathrm{Al}}^2}{\sigma_{\mathrm{W}} \pi r_{\mathrm{W}}^2} = E_{\mathrm{Al}} \frac{2\sigma_{\mathrm{W}} \pi r_{\mathrm{Al}}^2}{\sigma_{\mathrm{W}} \pi \left(2 r_{\mathrm{Al}}\right)^2} \qquad \Rightarrow \qquad E_{\mathrm{W}} = \frac{E_{\mathrm{Al}}}{2}$$

6. (8 points) When a battery with emf $\mathcal{E}$ is connected to the capacitor as shown on the left, a charge Q is found on the positive plate. If a material with dielectric constant κ is inserted to fill the gap while the battery remains connected as shown on the right, what charge will now be found on the positive plate?

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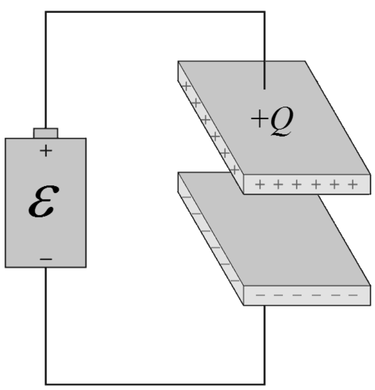
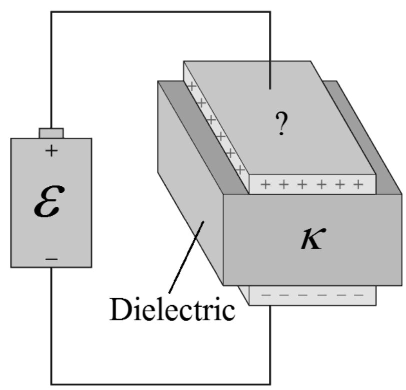




An ideal parallel-plate capacitor with vacuum between the plates has capacitance $C = \epsilon_0 A/d$, so, for the dimensions to be correct, every capacitor with vacuum between the plates must have a capacitance that is proportional to the permittivity of free space, ϵ_0. If the material between the plates isn't "free space" (vacuum), then the capacitance must be proportional to the permittivity of the material, ϵ. As $\epsilon = \kappa\epsilon_0$, the capacitance must increase by a factor of κ when the dielectric is inserted. From the definition of capacitance, $Q = C\,\Delta V$, the new charge must be

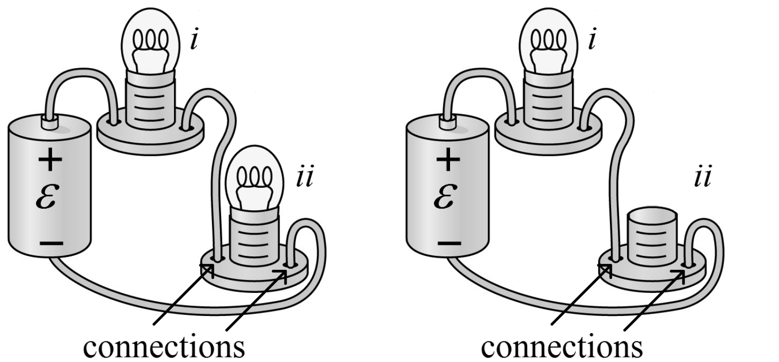
$$\kappa Q$$

7. (8 points) A battery with emf $\mathcal{E}$ is connected to two identical bulbs, i and ii, as shown on the left. Then bulb ii is removed from its socket, as shown on the right. What is the potential difference between the connections to bulb ii's socket in each case?

Apply Kirchhoff's Loop Rule, $\mathcal{E} - \Delta V_{ii} - \Delta V_i = 0$.

The bulbs are identical, so $\Delta V_i = \Delta V_{ii}$ when bulb ii is in place. Then

$$\mathcal{E} - \Delta V_{ii} - \Delta V_{ii} = 0 \qquad \Rightarrow \qquad 2\Delta V_{ii} = \mathcal{E} \qquad \Rightarrow \qquad \Delta V_{ii} = \mathcal{E}/2$$



No current can flow when bulb ii is removed, so $\Delta V_i = IR = 0$. Then

$$\mathcal{E} - \Delta V_{ii} - \Delta V_{ii} = 0 \qquad \Rightarrow \qquad \mathcal{E} - \Delta V_{ii} - 0 = 0 \qquad \Rightarrow \qquad \Delta V_{ii} = \mathcal{E}$$

So the potential difference between the connections to bulb ii's socket is

$\mathcal{E}/2$ with bulb ii in place, $\mathcal{E}$ with bulb ii removed.