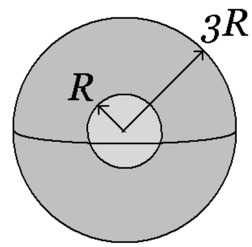
I. (16 points) A hollow insulating *sphere* has uniform volume charge density ρ, inner radius R, and outer radius $3R$. Find the magnitude of the electric field at a distance $2R$ from the center of the sphere. Express your answer in terms of parameters defined in the problem, and physical or mathematical constants.



. . . .

Use Gauss' Law, $\epsilon_0\Phi = q_{\text{in}}$. Let us first find the electric flux. Choose a Gaussian Surface with the symmetry of the charge distribution and passing through the point at which the electric field will be found. This is a sphere with radius $2R$, centered at the center of the charge distribution.

$$\Phi = \oint \vec{E}\cdot d\vec{A} = E4\pi\,(2R)^2 = 16\pi R^2 E$$

Next, find the charge *within* the Gaussian Surface. Since the volume charge density is uniform

$$\rho = \frac{q}{V_{\text{ol}}} \qquad \Rightarrow \qquad q_{\text{in}} = \rho V_{\text{ol}} = \rho\left[\frac{4}{3}\pi\,(2R)^3 - \frac{4}{3}\pi\,(R)^3\right] = \frac{4}{3}\pi\rho\left(7R^3\right)$$

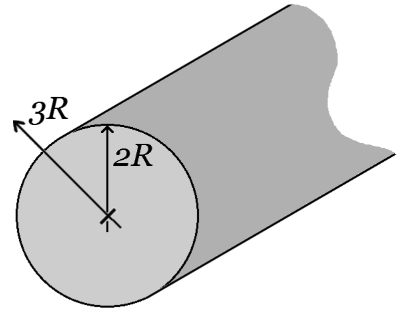
Putting these together

$$\epsilon_0\Phi = q_{\text{in}} \qquad \Rightarrow \qquad \epsilon_0 16\pi R^2 E = \frac{4}{3}\pi\rho\left(7R^3\right) \qquad \Rightarrow \qquad E = \frac{7\rho R}{12\epsilon_0}$$

II. (16 points) An infinite solid insulating cylinder has radius $2R$, as illustrated. Its volume charge density, ρ, varies with distance r from the center according to

$$\rho = \rho_0 \frac{R}{r}$$

where ρ_0 is a positive constant. Find the electric field magnitude at a distance $3R$ from the center in terms of parameters defined in the problem, and physical or mathematical constants.



. . . .

Use Gauss' Law, $\epsilon_0 \Phi_E = \epsilon_0 \oint \vec{E} \cdot d\vec{A} = q_{\text{in}}$. Choose a surface that passes through the point at which the electric field is to be found, and with the same symmetry as the charge distribution. A finite cylinder of radius $3R$ and length L satisfies these conditions.

Note that the flux through the ends of the Gaussian Surface is zero, as the electric field vectors are perpendicular to the outward-pointing area vectors. The electric field has constant magnitude over the curved side of the Gaussian Surface, and is parallel to the outward-pointing area vectors.

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \oint E \cos\theta \, dA = E \cos(0°) \oint dA = EA = E2\pi r L = E2\pi (3R) L = E6\pi R L$$

The charge inside the Gaussian Surface can be found from the volume charge density, as $\rho = dq/dV$. Choose a thin cylindrical shell for the volume element, $dV = 2\pi r L \, dr$.

$$q_{\text{in}} = \int \rho \, dV = \int_0^{2R} \rho_0 \frac{R}{r} 2\pi r L \, dr = \rho_0 R 2\pi L \int_0^{2R} dr$$

$$= \rho_0 R 2\pi L r \Big|_0^{2R} = \rho_0 R 2\pi L (2R - 0) = \rho_0 4\pi R^2 L$$

So

$$\epsilon_0 \Phi_E = q_{\text{in}} \quad \Rightarrow \quad \epsilon_0 E 6\pi R L = \rho_0 4\pi R^2 L \quad \Rightarrow \quad E = \frac{2\rho_0 R}{3\epsilon_0}$$

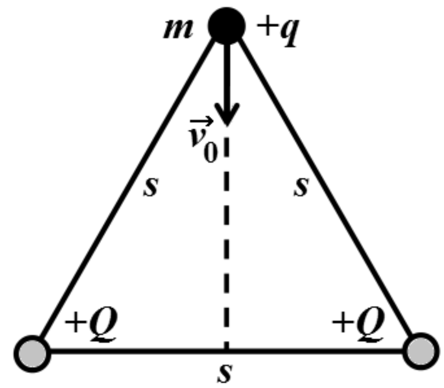
1. (6 points) In the problem above, what is the direction of the electric field at a distance $3R$ from the center?

. . . .

Since ρ_0 is positive, $\rho = \rho_0 R/r$ is also positive, and the hollow cylinder is positively charged. Electric field vectors point away from positive charge.

Away from the center.

III. (16 points) Two positive charges $+Q$ are fixed at the vertices of an equilateral triangle with sides of length s. A particle of positive charge $+q$ and mass m is positioned at the apex of the equilateral triangle as shown. It is launched from that point with an initial velocity $\vec{v}_0$ along the center line as shown. What must the minimum initial speed v_0 of this particle be so that it passes between the two fixed charges? Express your answer in terms of parameters defined in the problem and physical or mathematical constants. (*NOT on Earth—no gravity!*)



. . . .

Use the Work-Energy Theorem,

$$W_{\text{ext}} = \Delta K + \Delta U + \Delta E_{\text{th}}$$

Choose a system consisting of all three charged particles. There is no work done by external forces on that system, and there are no non-conservative forces converting mechanical energy to thermal energy within that system. The potential energy is that of a system of charged particles.

$$0 = (K_f - K_i) + (U_f - U_i) + 0$$

So

$$0 = \left(\tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2\right) + \left(K\frac{Qq}{r_f} - K\frac{Qq}{r_i}\right) + \left(K\frac{Qq}{r_f} - K\frac{Qq}{r_i}\right) + \left(K\frac{Q^2}{r_f} - K\frac{Q^2}{r_i}\right)$$

where the potential energy of the system is the sum of the potential energies associated with each pair of particles. Note that the final speed of the particle with charge q must be only infinitesimally more than zero as it passes between the particles with charge Q.

$$0 = \left(0 - \tfrac{1}{2}mv_0^2\right) + \left(K\frac{Qq}{s/2} - K\frac{Qq}{s}\right) + \left(K\frac{Qq}{s/2} - K\frac{Qq}{s}\right) + \left(K\frac{Q^2}{s} - K\frac{Q^2}{s}\right)$$

Solve for v_0:

$$\tfrac{1}{2}mv_0^2 = 2\left(K\frac{Qq}{s/2} - K\frac{Qq}{s}\right) + 0 = 2K\frac{Qq}{s} \qquad \Rightarrow \qquad v_0 = \sqrt{\frac{4K}{m}\frac{Qq}{s}}$$

2. (6 points) Consider a situation in which the particle with charge q in the problem above were replaced by a particle with charge $q' = 2q$, and the fixed charges Q were each replaced with fixed charges $Q' = 2Q$. How does the minimum speed, v_0', required for the particle to pass the fixed charges in this situation, compare to your answer v_0 above?

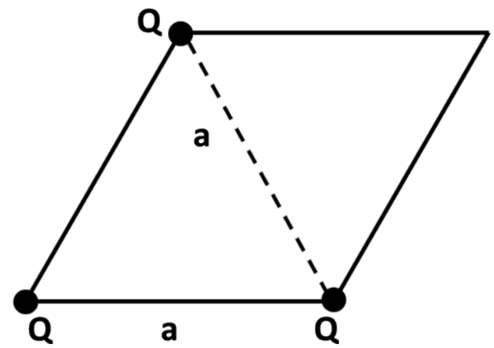
. . . .

Since the electric potential energy depends on the product of the pairs of charges, doubling all the charge values will increase the potential energy by a factor of 4. Initially, then, the moving particle will need four times the kinetic energy to pass between the fixed charges. As the kinetic energy depends on the square of the speed, doubling the speed will provide four times the kinetic energy.

$$v_0' = 2v_0$$

3. (8 points) Three particles, each with charge Q, are located as shown on different corners of a rhombus with sides of lengths a and a diagonal of length a (a rhombus has 4 equal length sides that do not intersect at right angles). With respect to zero at infinity, what is the electric potential at the empty vertex?

. . . .

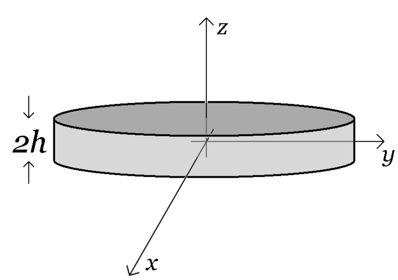


Since electric potential is a scalar, the potential at the empty vertex is just the sum of the potentials contributed by each of the three charges Q. With respect to zero at infinity, the electric potential due to a single point charge is $V = Kq/r$. Each of the charges Q at the ends of the diagonal with length a are a distance a from the empty vertex, and so each contributes electric potential KQ/a. The remaining charge is a distance $a\cos(30°) = a\sqrt{3}/2$ from the diagonal of length a, and so is twice that distance, or $a\sqrt{3}$ from the empty vertex. Therefore, it contributes electric potential $KQ/a\sqrt{3}$. The total potential at the empty vertex is the sum

$$K\frac{Q}{a} + K\frac{Q}{a} + K\frac{Q}{a\sqrt{3}} = \left(2 + \frac{1}{\sqrt{3}}\right) K\frac{Q}{a}$$

4. (8 points) An infinite slab with thickness $2h$ has uniform volume charge density ρ. The slab is infinite in the x and y directions and centered at the origin, extending from $-h$ to $+h$ along the z axis. A finite segment of the slab is illustrated. Are there any locations where the magnitude of the electric field is zero, and if so, where?

. . . .

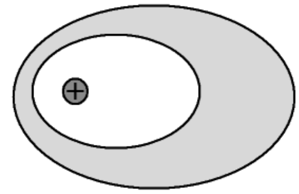


Consider the symmetry of the slab. If it is, for example, rotated 180° about the x axis, the charge distribution is unchanged. The electric field must, therefore, also be unchanged. The 180° rotation about the x axis would reverse the direction of any field on the x−y plane. The only way an electric field could be both reversed and unchanged is if its magnitude were zero.

Yes, the field has zero magnitude only on the x−y plane, $z = 0$.

5. (8 points) A hollow conductor, illustrated in cross-section, carries a net charge of -3 nC. Within its void lies a particle with a charge of +5 nC. What is the net charge on the inner and outer surfaces of the conductor at equilibrium?

. . . .



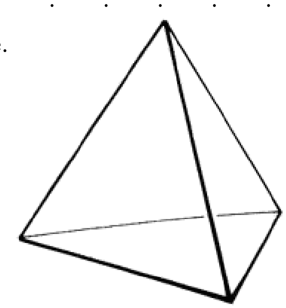
Consider a Gaussian Surface within the solid part of the conductor. The field in the solid part of a conductor at equilibrium is zero, so the flux through this Gaussian Surface is zero, so the net charge contained within the Gaussian Surface must be zero. There must be a charge of -5 nC on the inner surface of the conductor to balance the +5 nC charge on the particle.

Charge is conserved. If there is -5 nC on the inner surface of the conductor, but the conductor has a net charge of -3 nC, then there must be +2 nC on the outer surface.

$$Q_{\text{inner}} = -5\,\text{nC while } Q_{\text{outer}} = +2\,\text{nC}$$

6. (8 points) A positive point charge $+q$ lies at the center of a tetrahedron, constructed of four equilateral triangles with edges a. What is the electric flux through the bottom face of the tetrahedron?

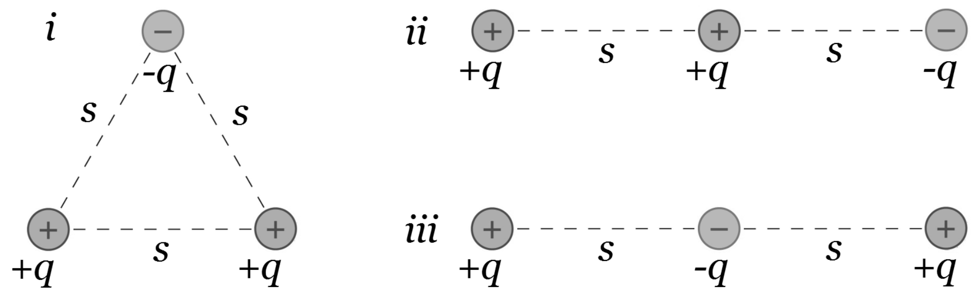
. . . .



Each face is identical, so one-fourth of the total flux must pass through each face.
Use Gauss' Law.

$$\epsilon_0 \Phi = q_{\text{in}} \qquad \Rightarrow \qquad \Phi/4 = +q/4\epsilon_0$$

7. (8 points) Three isolated systems, i, ii, and iii, each consist of a negatively charged particle $-q$ and two positively charged particles $+q$, all with the same charge magnitude. Let the configuration with zero electric potential energy be the same for each system. Rank the electric potential energies U of the systems from greatest to least. (Remember that since the systems are isolated, there is no interaction between them.)



. . . .

Recall that the electric potential energy of a system of two point charges is $U = Kq_1q_2/r$ if infinite separation is chosen as the zero point, and that the electric potential energy of a system of multiple point charges is just the sum of the electric potential energies of all the pairs.

Compare system i and system ii. In system ii, the two positively charged particles are the same distance apart as in system i, so there is no electric potential energy difference due to that interaction. The negatively charged particle is the same distance from one of the positively charged particles, but it is twice as far from the other one. Work would have to be done on system i to convert it to system ii, as the negatively charged particle is attracted to the positively charged particle is it being pulled farther from. So, $U_{ii} > U_i$.

Next, compare system i and system iii. In system iii, the negatively charged particle is the same distance from the positively charged particles as in system i, so there is no electric potential energy difference due to those interactions. The positively charged particles, however, are twice as far apart. Energy would be released as system i is converted to system iii, as the two positively charged particles repel each other. So, $U_{iii} > U_i$.

$$U_{ii} > U_i > U_{iii}$$