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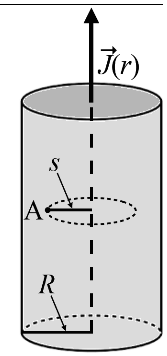
*I*. (16 points) An infinitely long straight wire of radius $R$ carries a non-uniform current density

$$\vec{J}(r) = J_0 \left(\frac{R}{r}\right) \hat{k}$$

distributed along its cross-section, where $r$ is the distance from the center of the wire. The value $J_0$ is the current density magnitude at the surface of the wire.

Determine the **magnitude and direction** of the magnetic field at the point $A$ that is a distance $s$ away from the wire center and inside the wire. Express your answer in terms of the parameters defined in the problem and physical or mathematical constants.

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Use Ampere's Law. Choose a circular Amperian Loop of radius $s$.

$$\oint \vec{B} \cdot d\vec{s} = \oint B \cos\theta \, ds = \mu_0 I_{\text{thru}}$$

The magnitude of the field is uniform around this loop, and the direction is tangent to the loop, so $\theta$ is everywhere zero.

$$B \oint \cos(0) \; ds = B \oint ds = B \, 2\pi \, s = \mu_0 \int \vec{J} \cdot d\vec{A}$$

The current density varies with $r$, so choose an area element $d\vec{A}$ that is "small" in the $r$ direction. A thin ring of circumference $2\pi \, r$ and width $dr$ is suitable. Let the direction of this area element be parallel to $\vec{J}$.

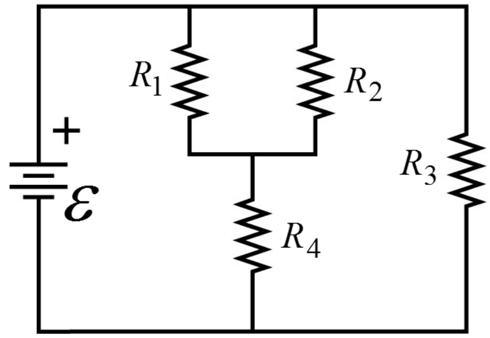
$$B \, 2\pi \, s = \mu_0 \int J \cos\phi \, dA = \mu_0 \int_0^s J_0 \left(\frac{R}{r}\right) \cos(0) \; 2\pi \, r \, dr$$

$$= 2\pi\mu_0 J_0 R \int_0^s dr = 2\pi\mu_0 J_0 R \, r \Big|_0^s = 2\pi\mu_0 J_0 R \Big[s - 0\Big] = 2\pi\mu_0 J_0 Rs$$

where the limits were set to calculate all the current the passes through the Amperian Loop of radius $s$. Solve for $B$. The direction can be determined by a Right Hand Rule.

$$B \, 2\pi \, s = 2\pi\mu_0 J_0 Rs \qquad \Rightarrow \qquad \vec{B} = \mu_0 J_0 R \text{ out of the page}$$

*II.* (16 points) The battery in the circuit shown has emf $\mathcal{E} = 24\,\mathrm{V}$. The resistances are $R_1 = 4.0\,\Omega$, $R_2 = 12\,\Omega$, $R_3 = 8.0\,\Omega$, and $R_4 = 5.0\,\Omega$. What is the current through resistor $R_2$?



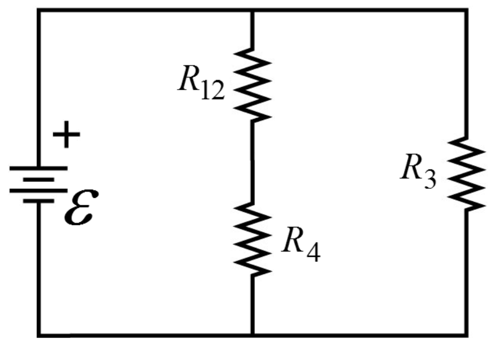

Combine resistors in series or parallel.

$R_1$ and $R_2$ are in parallel (same potential difference), so

$$R_{12} = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-1} = \left(\frac{1}{4.0\,\Omega} + \frac{1}{12\,\Omega}\right)^{-1} = 3.0\,\Omega$$

$R_{12}$ and $R_4$ are in series (same current), so

$$R_{124} = R_{12} + R_4 = 3.0\,\Omega + 5.0\,\Omega = 8.0\,\Omega$$




The current through $R_{124}$ can be determined from the definition of resistance,

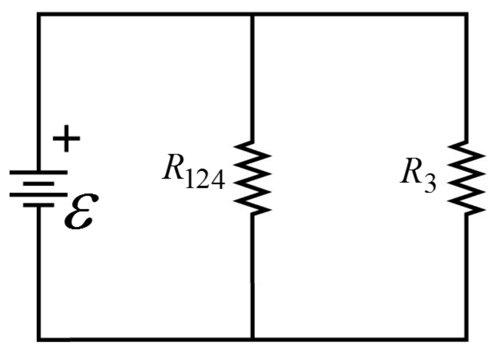
$$\Delta V_{124} = I_{124} R_{124} \qquad \Rightarrow \qquad I_{124} = \frac{\mathcal{E}}{R_{124}} = \frac{24\,\mathrm{V}}{8.0\,\Omega} = 3.0\,\mathrm{A}$$

Because $R_{12}$ and $R_4$ are in series, this must also be the current through those resistances.

$$I_{124} = I_{12} = I_4$$

The potential across $R_{12}$ can be determined from the definition of resistance,

$$\Delta V_{12} = I_{12} R_{12} = (3.0\,\mathrm{A})\,(3.0\,\Omega) = 9.0\,\mathrm{V}$$




Because $R_1$ and $R_2$ are in parallel, this must also be the potential across those resistances.

$$\Delta V_{12} = \Delta V_1 = \Delta V_2$$

The current through $R_2$ can be determined from the definition of resistance,

$$\Delta V_2 = I_2 R_2 \qquad \Rightarrow \qquad I_2 = \frac{\Delta V_2}{R_{124}} = \frac{9.0\,\mathrm{V}}{12\,\Omega} = 0.75\,\mathrm{A}$$

1. (6 points) In the problem above, compare the power $P_1$ dissipated in resistor $R_1$ to the power $P_2$ dissipated in resistor $R_2$.

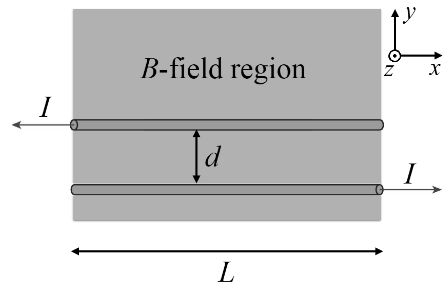
Electric power is $P = I\,\Delta V$. Resistances $R_1$ and $R_2$ are in parallel, so they have the same potential across them, but may have different currents through them. Substituting the definition of resistance to eliminate the current,

$$P = \left(\frac{\Delta V}{R}\right) R = \frac{(\Delta V)^2}{R}$$

we see that, for resistors with a given potential difference, the power dissipated is inversely proportional to the resistance. $R_1$ has one-third the resistance of $R_2$, so it must dissipate three times the power.

$$P_1 = 3P_2$$

*III*. (16 points) Two identical current-carrying wires with mass $m$ and length $L$ have been placed in uniform background magnetic field with magnitude $B_0$ directed out of the page. The same current $I$ flows through each wire as shown. If separation of the wires is $d$, find the **magnitude and direction** of the net force acting on the bottom wire. Express your answer in terms of the parameters defined in the problem and physical or mathematical constants. (*In space, far from any other electric, magnetic, or gravitational fields.*)

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The Bio-Savart Law for currents, $d\vec{B} = \frac{\mu_0}{4\pi}\frac{d\vec{\ell}\times\hat{r}}{r^2}$, and the Right Hand Rule tell us that the field due to the top wire is out of the page at the location of the bottom wire. Since that's the same direction as the external field, the total field at the location of the bottom wire is out of the page, and has a magnitude equal to the scalar sum of the two contributing magnetic fields.

Assuming that $L$ is much greater than $d$, the field due to the top wire at the location of the bottom wire can be found using Ampere's Law. Choose a path that follows a circular field line of radius $d$.

$$\oint \vec{B}\cdot d\vec{s} = \mu_0 I_{\text{thru}} \qquad \Rightarrow \qquad B_{\text{top}}\, 2\pi d = \mu_0 I \qquad \Rightarrow \qquad B_{\text{top}} = \frac{\mu_0 I}{2\pi d}$$

so the total field out of the page at the location of the bottom wire is

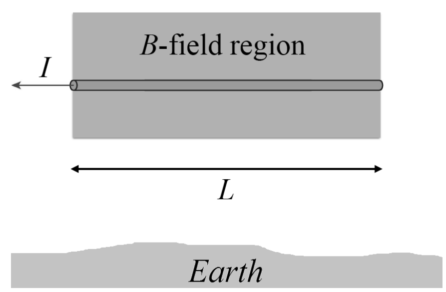
$$B_{\text{toal}} = B_0 + B_{\text{top}} = B_0 + \frac{\mu_0 I}{2\pi d}$$

Then the force magnitude on the bottom wire in this magnetic field can be found using

$$\vec{F} = I\vec{\ell}\times\vec{B} \qquad \Rightarrow \qquad F = IL\left(B_0 + \frac{\mu_0 I}{2\pi d}\right)\sin\left(90^\circ\right)$$

and direction is determined by the Right Hand Rule. So

$$F = IL\left(B_0 + \tfrac{\mu_0 I}{2\pi d}\right) \text{ down the page}$$




2. (6 points) A single wire of length $L$ carrying current $I$ is immersed in the uniform magnetic field of unknown magnitude and direction, as shown. If the mass of the current-carrying wire is $m$, what minimum magnetic field strength and direction will levitate the wire? (*On Earth—you may neglect the Earth's magnetic field, but do NOT neglect gravity.*)

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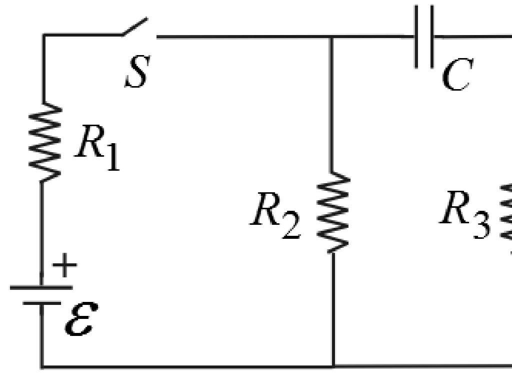
For the wire to levitate, the net force must be zero, so the magnetic force must be up the page, opposite the gravitational force. As the magnetic force on a wire is

$$\vec{F} = I\vec{\ell}\times\vec{B}$$

the Right Hand Rule tells us the magnetic field must be

$$B = mg/IL, \text{ directed out of the page}$$

3. (8 points) In the circuit shown, the emf $\mathcal{E} = 24\,\mathrm{V}$ and the capacitance $C = 3\,\mu\mathrm{F}$. The resistances are $R_1 = 4\,\Omega$, $R_2 = 8\,\Omega$, and $R_3 = 20\,\Omega$. After the switch $S$ has been closed for a very long time, what is the charge on the capacitor?

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After the switch has been closed for a "very long" time, the capacitor will be fully charged. Therefore, there can be no current flowing in the branch with the capacitor (or else charge would be delivered to it, and it wouldn't be fully charged). With no current flowing in the branch with the capacitor, there is no current flowing through resistor $R_3$, so the potential across $R_3$ is zero, and the potential across the capacitor must be the same as that across resistor $R_2$.

The currents through resistors $R_1$ and $R_2$ can be found by applying Kirchhoff's Loop Rule to the left loop. Note that the currents must be the same if no current flows into the branch with the capacitor. Traveling clockwise,

$$+\mathcal{E} - IR_1 - IR_2 = 0 \qquad \Rightarrow \qquad I = \frac{\mathcal{E}}{R_1 + R_2} = \frac{24\,\mathrm{V}}{4\,\Omega + 8\,\Omega} = 2\,\mathrm{A}$$

The definition of resistance can be used to determine the potential difference across resistor $R_2$.

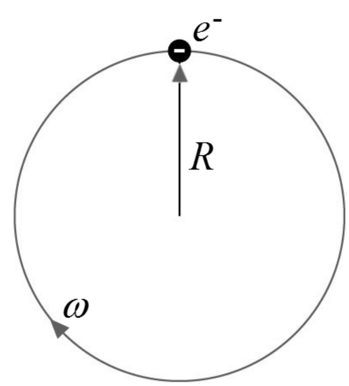
$$\Delta V_2 = I_2 R_2 = (2\,\mathrm{A})\,(8\,\Omega) = 16\,\mathrm{V}$$

which we've noted must be the same as the potential across the capacitor.

Then the definition of capacitance can now be used to determine the charge on the capacitor.

$$Q = C\,\Delta V = (3\,\mu\mathrm{F})\,(16\,\mathrm{V}) = 48\,\mu\mathrm{C}$$

4. (8 points) An electron moves in a uniform magnetic field. Its circular path has radius $R$, and it has clockwise angular velocity $\omega$. What is the direction of the magnetic field?

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As the electron is moving in a circle, its acceleration, and therefore the net force on it, must be toward the center of the circle. At the instant shown, this magnetic force $\vec{F} = q\vec{v} \times \vec{B}$ must be down the page. Because the electron has negative charge, $\vec{v} \times \vec{B}$ must, therefore, be up the page at this instant. With $\vec{v}$ to the right, the Right Hand Rule shows us that the magnetic field is

Into the page.

5. (8 points) In the circuit shown, the positive direction of current flow through each resistor has been defined as indicated with arrows. If the current through resistor $R_1$ is $I_1$, etc., which of the following is a valid expression of Kirchhoff's Loop Law?

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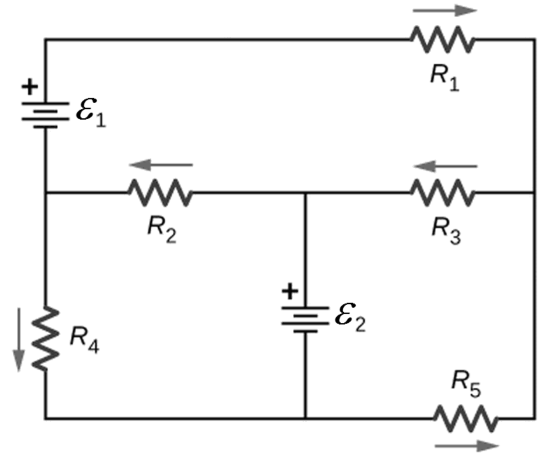


With the positive current direction through each resistor as shown, the electric potential decreases when traveling with the positive current direction, and increases when traveling against it. The only expression of Kirchhoff's Loop Law offered that both represents a closed path and has all these potential changes correct is

$$+\mathcal{E}_1 - I_1R_1 + I_5R_5 + \mathcal{E}2 - I_2R_2 = 0$$

6. (8 points) What is the line integral of the vector $\vec{B}$ along the loop shown in the figure below? The magnetic field is produced by the wire carrying a current $I$, located at the center of the loop's arc and oriented perpendicular to it. The direction of the current and integration loop are as illustrated in the figure.

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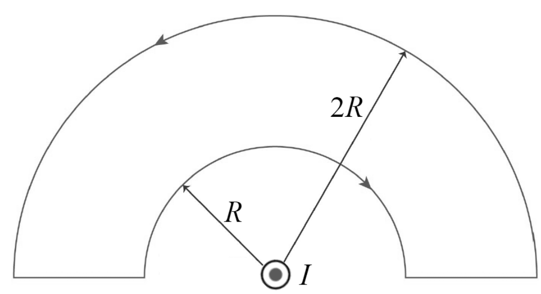


Ampere's Law relates the closed path integral of the magnetic field around a loop to the current that passes through the loop.

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{\text{thru}}$$

In this particular case, the current doesn't pass through the loop. $I_{\text{thru}}$ is zero, so $\oint \vec{B} \cdot d\vec{s}$ is also

Zero

7. (8 points) A small circular coil of wire (seen edge-on in a cutaway view) is between the poles of a magnet. The coil carries a current in the direction indicated. What will be the effect of the magnetic field on the coil at the instant it is positioned as shown? The coil will experience . . .

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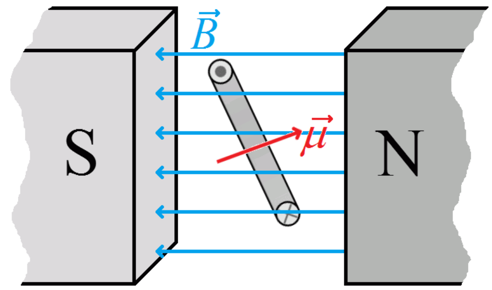


The magnetic field, $\vec{B}$, due to the poles points to the left, from North to South. The magnetic force on each portion of the loop is balanced by a magnetic force on the portion direction opposite, so there is no net force on the loop. However, these forces do not have the same line of action, so there will be a net torque.

The Right Hand Rule shows that the magnetic moment, $\vec{\mu}$, of the loop is up and to the right. The torque on a current loop in a magnetic field is

$$\vec{\tau} = \vec{\mu} \times \vec{B}$$

The Right Hand Rule shows that there is . . . a net torque directed out of the page.