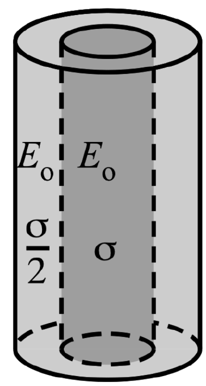

I. (16 points) A tungsten-clad aluminum wire consists of a solid aluminum core (having conductivity σ) of radius R, which is then covered in a tungsten sheath (having conductivity $\sigma/2$) that extends to radius $2R$. When a potential difference is applied across the ends of the wire, an internal field of magnitude E_0 is induced within both the core and the sheath. What is the total current that will be flowing in the wire, under these circumstances? Express your answer in terms of the parameters defined in the problem, and physical or mathematical constants.



. . . .

The current in the wire is the sum of the current in the aluminum core, I_{Al}, and the current in the tungsten cladding, I_{W}. Current can be related to current density, which can, in turn, be related to conductivity and electric field. In terms of magnitudes,

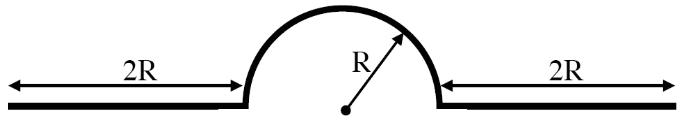
$$J = \sigma E = I/A \qquad \Rightarrow \qquad I = \sigma E A$$

So

$$I_{\text{total}} = I_{\text{Al}} + I_{\text{W}} = \sigma_{\text{Al}} E_{\text{Al}} A_{\text{Al}} + \sigma_{\text{W}} E_{\text{W}} A_{\text{W}}$$

$$= \sigma E_0 \left[\pi R^2\right] + \frac{\sigma}{2} E_0 \left[\pi (2R)^2 - \pi R^2\right] = \sigma E_0 \pi R^2 + \frac{3}{2}\sigma E_0 \pi R^2 = \frac{5}{2}\sigma E_0 \pi R^2$$

II. (16 points) The wire in the figure has uniform linear charge density λ. What is the electric potential, with respect to zero at infinite distance, at the center of the semicircle? Express your answer in terms of the parameters defined in the problem, and physical or mathematical constants.



. . . .

Potential is a scalar, so the potential due to this wire is the sum of the potential due to the curved part, V_c, and twice the potential due to one straight part, V_s.

Since every piece of the curved part is the same distance from the center, the potential due to this part is the same as that of a point particle with the same charge at the same distance.

$$V_c = K\frac{Q_c}{R_c} = K\frac{\lambda L}{R} = K\frac{\lambda\pi R}{R} = K\lambda\pi$$

Each piece of the straight part, however, is a different distance from the center. Each small element of the straight part is like a point. Adding up the contributions of all these small elements of charge (taking an integral) yields the potential due to one entire straight part.

$$V_s = \int dV_s = \int K\frac{dq}{r} = \int_R^{3R} K\frac{\lambda\,dr}{r} = K\lambda\ln r\Big|_R^{3R} = K\lambda\ln\left(\frac{3R}{R}\right) = K\lambda\ln 3$$

So

$$V = V_c + 2V_s = K\lambda\pi + 2\left(K\lambda\ln 3\right) = K\lambda\left(\pi + 2\ln 3\right)$$

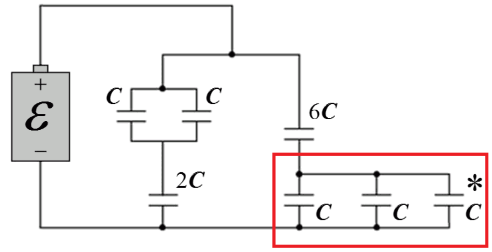
1. (6 points) In the problem above what is the direction of the electric potential at the center of the semicircle?

. . . .

Potential is a scalar, so

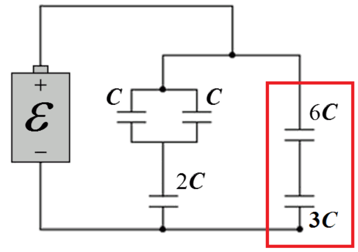
This is not a meaningful question.

III. (16 points) A battery with emf $\mathcal{E}$ is connected to a network of seven capacitors (five with capacitance C, one with capacitance $2C$, and one with capacitance $6C$) as shown. With respect to zero in the uncharged state, what energy is stored in the capacitor farthest to the right, marked with an asterisk? Express your answer in terms of the parameters defined in the problem, and physical or mathematical constants.



. . . .

Combine capacitors in series or parallel, as appropriate.

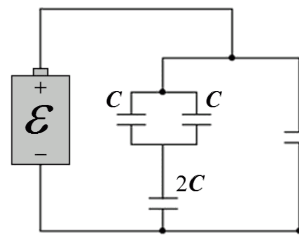


Three capacitors in the top figure, including the one marked with an asterisk, are in parallel as they have the same potential difference across them. In parallel,

$$C_{\text{eq}} = \sum C_i = C + C + C = 3C$$

In the middle figure, these three capacitors have been replaced by their $3C$ equivalent capacitance. The two capacitors on the right of the middle figure are in series, as they'll have the same charge. In series

$$C_{\text{eq}} = \left(\sum \frac{1}{C_i}\right)^{-1} = \left(\frac{1}{6C} + \frac{1}{3C}\right)^{-1} = \left(\frac{1}{6C} + \frac{2}{6C}\right)^{-1} = 2C$$



In the bottom figure, these two capacitors have been replaced by their $2C$ equivalent capacitance, and it can now be seen that the potential across this $2C$ equivalent capacitance is $\mathcal{E}$. From the definition of capacitance, $Q = C\,\Delta V$, the charge on the $2C$ equivalent capacitance is

$$Q_{2C} = (2C)\,\mathcal{E}$$

In the middle figure, since the $6C$ and $3C$ capacitors are in series, they must each also have that same charge $2C\mathcal{E}$. The potential across the $3C$ equivalent capacitance is

$$\Delta V_{3C} = \frac{Q_{3C}}{3C} = \frac{2C\mathcal{E}}{3C} = \frac{2}{3}\mathcal{E}$$

In the top figure, since the three capacitors C are in parallel, each, including the one marked with an asterisk, must have that same potential $2\mathcal{E}/3$. The potential energy stored in the capacitor marked with an asterisk, therefore, is

$$U = \frac{1}{2}C\,(\Delta V)^2 = \frac{1}{2}C\left(\frac{2}{3}\mathcal{E}\right)^2 = \frac{2}{9}C\mathcal{E}^2$$

2. (6 points) In the problem above, let your answer (the energy stored in the capacitor farthest to the right) be U_0. If the battery with emf $\mathcal{E}$ is replaced by a battery with emf $2\mathcal{E}$, what energy U' will now be stored in that capacitor?

. . . .

As the only source of potential in the circuit is the battery, the potential across *every* capacitor in this circuit must be proportional to $\mathcal{E}$, if only to make the dimensions correct. The potential energy stored in a capacitor is $U = \frac{1}{2}C\,(\Delta V)^2$, so the potential energy stored in *every* capacitor in this circuit must be proportional to $\mathcal{E}^2$. If the emf is doubled, the energy stored in *every* capacitor in this circuit must increase by a factor of four.

$$U' = 4U_0$$

3. (8 points) An electric field varies with position x, as shown. At what point does the electric potential have its maximum value? (Remember, potential is a signed scalar.)

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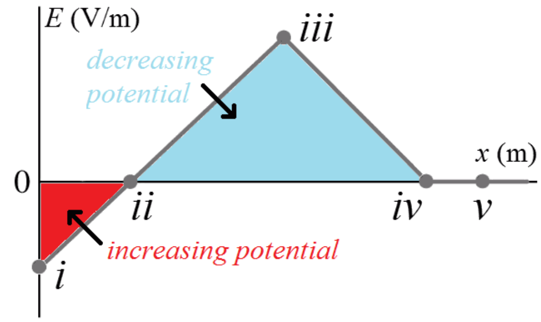
As potential and field are related by

$$\Delta V = -\int \vec{E} \cdot d\vec{s}$$



potential increases as negative area under the curve is accumulated while moving in the x direction, and potential decreases as positive area is accumulated. The maximum potential change from the starting point (and thus the maximum potential) occurs when the potential stops increasing and begins decreasing. In this case, that is at

Point ii.

4. (8 points) The electric potential varies with position x, as shown. At what point does the electric field magnitude have its maximum value?

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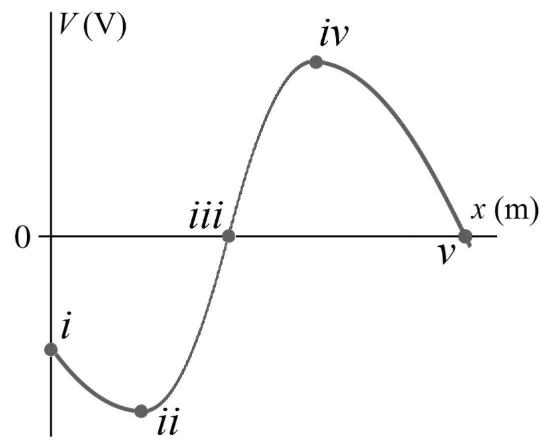
As potential and field are related by

$$E_x = -\frac{\delta V}{\delta x}$$



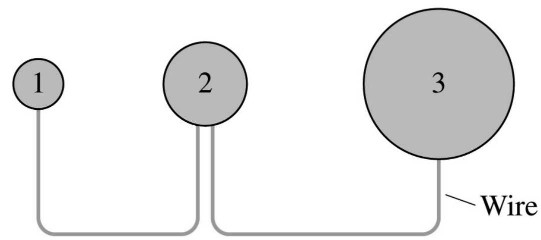
the electric field magnitude is greatest where the slope of the graph is steepest, which is

Point iii.

5. (8 points) Three charged metal spheres of different radii are connected by a thin metal wire. Choose the expression that best describes the relationships among the potentials, charge, and electric fields at the surface of each sphere. The spheres are sufficiently far apart that they do not affect each other's charge distribution.

. . . .

The wires make this one big conductor at equilibrium, and the potential is the same everywhere in a conductor at equilibrium. The charge on each sphere creates a potential at its surface as if the charge were concentrated at its center. As $V = KQ/R$ in that situation, spheres of different radius but the same potential must have different charges. Only one of the offered choices has equal potentials but different charges.



$V_1 = V_2 = V_3$ and $Q_1 < Q_2 < Q_3$ and $E_1 > E_2 > E_3$

6. (8 points) Currents in four wires are shown. What are the magnitude and the direction of the current in the fifth (bottom) wire?

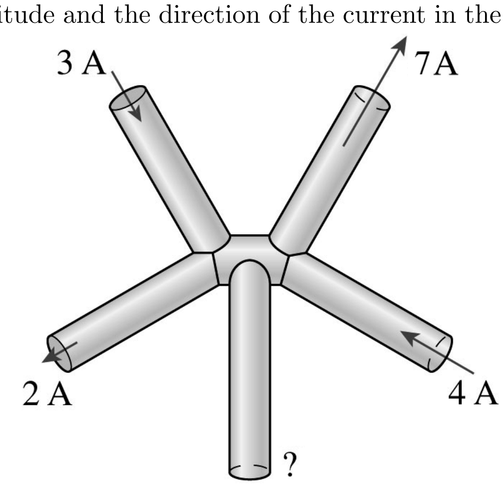
. . . .

Charge is conserved, so Kirchhoff's Junction (or Node) Law states that the current into a junction must equal the current out of a junction. One way to express this is to let current into a junction be positive, and that out be negative, so the total must be zero. If the unknown current is X, then

$$4\,\mathrm{A} - 7\,\mathrm{A} + 3\,\mathrm{A} - 2\,\mathrm{A} + X = 0$$

so

$$X = -4\,\mathrm{A} + 7\,\mathrm{A} - 3\,\mathrm{A} + 2\,\mathrm{A} = +2\,\mathrm{A}$$



Since we let current into the junction be positive, this represents

2 A into the junction

7. (8 points) A parallel-plate capacitor is connected to a battery, allowed to charge, then disconnected. Insulating handles are then used to push the plates closer together, until they are only half as far apart as they were originally. How does the energy now stored in the capacitor (with respect to zero in the uncharged state) compare to that stored originally?

. . . .

If the battery is disconnected, then the charge on the capacitor can't change, but the potential can.

$$U = \frac{Q^2}{2C} = \frac{Q^2}{2\epsilon_0 A/d} = d\frac{Q^2}{2\epsilon_0 A}$$

shows that in this circumstance, the potential energy stored in the capacitor is proportional to the spacing d between the plates.

The new energy is **half as much** as the original energy.