
I. (16 points) An electron (mass m_e, charge $-e$) is at rest on the surface of a fixed uniform insulating sphere with positive charge Q and radius R. The electron is then given a speed v_0 away from the sphere. What maximum distance from the center of the sphere will the electron reach? Express your answer in terms of the parameters defined in the problem, and physical or mathematical constants.

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Use the Work-Energy Theorem

$$W_{\text{ext}} = \Delta K + \Delta U + \Delta E_{\text{th}}$$

A system consisting of the sphere and the electron has no external forces on it, so external forces do no work. There is no non-conservative internal force transforming mechanical energy to thermal energy. As the sphere is fixed in place, only the electron can have kinetic energy. The potential energy is due to the internal conservative electric force.

$$0 = \left(\tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2\right) + \left(\frac{Kq_1q_2}{r_f} - \frac{Kq_1q_2}{r_i}\right)$$

The electron stops at its maximum distance ($v_f = 0$).

$$0 = \left(0 - \tfrac{1}{2}m_e v_0^2\right) + \left(\frac{KQ\,(-e)}{r_f} - \frac{KQ\,(-e)}{R}\right)$$

Solve for r_f.

$$\frac{KQe}{r_f} = \frac{KQe}{R} - \tfrac{1}{2}m_e v_0^2 \qquad \Rightarrow \qquad r_f = \frac{KQe}{\frac{KQe}{R} - \frac{1}{2}m_e v_0^2}$$

II. (16 points) A radially symmetric insulating sphere of charge has a radius R and a non-uniform volume charge density ρ that depends on distance r from the center according to

$$\rho = \rho_0 \left(\frac{r}{R}\right)^2$$

where ρ_0 is a positive constant. What is the magnitude of the electric field at a distance $2R/3$ from the center? Express your answer in terms of the parameters defined in the problem, and physical or mathematical constants.

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Use Gauss' Law, $\epsilon_0 \Phi = q_{\text{in}}$. Choose a spherical Gaussian Surface centered in the sphere, with a radius of $2R/3$, so it passes through the point at which the field is to be determined. Symmetry tells us that the field lines are perpendicular to this Gaussian Surface at every point, and that the magnitude of the field is the same at every point on it (that's why this surface was chosen). So

$$\Phi = \oint \vec{E} \cdot d\vec{A} = \oint E \cos\theta \, dA = E \cos 0 \oint dA = EA = E\, 4\pi r^2 = E\, 4\pi \left(\frac{2R}{3}\right)^2$$

Next, the charge inside the Gaussian Surface must be determined. As $\rho = dq/dV$,

$$q_{\text{in}} = \int dq = \int \rho \, dV$$

The volume charge density varies with r, so choose thin spherical shells for the volume element dV, as they are "small" in the r direction. Add up (integrate) all the charge from the center to the Gaussian Surface.

$$q_{\text{in}} = \int_0^{2R/3} \rho_0 \left(\frac{r}{R}\right)^2 4\pi r^2 \, dr = \frac{4\pi \rho_0}{R^2} \int_0^{2R/3} r^4 \, dr = \frac{4\pi\rho_0}{R^2} \left[\frac{r^5}{5}\right]_0^{2R/3} = \frac{4\pi\rho_0}{5R^2}\left[\frac{2R}{3}\right]^5$$

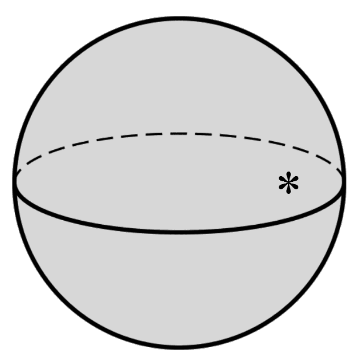
Putting the expressions for Φ and q_{in} together,

$$\epsilon_0 E\, 4\pi \left(\frac{2R}{3}\right)^2 = \frac{4\pi\rho_0}{5R^2}\left[\frac{2R}{3}\right]^5 \qquad \Rightarrow \qquad \epsilon_0 E = \frac{\rho_0}{5R^2}\left[\frac{2R}{3}\right]^3$$

So

$$E = \frac{\rho_0}{5\epsilon_0 R^2}\left[\frac{2R}{3}\right]^3 = \frac{\rho_0}{5\epsilon_0}\left[\frac{8R}{27}\right] = \frac{8\rho_0 R}{135\epsilon_0}$$

1. (6 points) The sphere in the problem above is shown at the right. What is the direction of the electric field at the point indicated with an asterisk, a distance $2R/3$ from the center?



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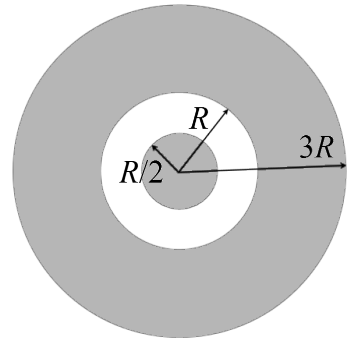
As ρ_0 is positive, a spherical Gaussian Surface centered in the sphere and passing through $2R/3$ will contain positive charge. It will, therefore, have positive flux through it, meaning that the field is directed

Away from the center.

III. (16 points) A positively charged insulating cylinder with radius $R/2$ has the uniform volume charge density ρ_0. The cylinder is placed at the center of a negatively charged insulating hollow cylinder with inner radius R, outer radius $3R$, and volume charge density $-\rho_0/4$. Calculate the magnitude of electric field at a distance $2R$ from the center of the cylinders. Express your answer in terms of the parameters defined in the problem, and physical or mathematical constants.



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Use Gauss' Law, $\epsilon_0 \Phi = q_{\text{in}}$. Choose a cylindrical Gaussian Surface of arbitrary length L centered on the charge distribution, with a radius of $2R$, so it passes through the point at which the field is to be determined. Symmetry tells us that the field lines are perpendicular to this Gaussian Surface at every point on the curved part of the Gaussian Surface, and that the magnitude of the field is the same at every point on that curved part. The field lines are tangent to the surface at the end caps, so the flux through them is zero.

$$\Phi = \oint \vec{E} \cdot d\vec{A} = \oint E \cos\theta \, dA = E \cos 0^\circ \oint dA_{\text{curved}} = EA_{\text{curved}} = E\, 2\pi r L = E\, 2\pi\, (2R)\, L = E\, 4\pi\, RL$$

Next, the charge inside the Gaussian Surface must be determined. As the charge density in each cylinder is uniform

$$q_{\text{in}} = \rho_{\text{in}} V_{\text{in}} + \rho_{\text{out}} V_{\text{out}} = \rho_0 \pi \left(\frac{R}{2}\right)^2 L + \frac{-\rho_0}{4} \left[\pi\, (2R)^2\, L - \pi\, (R)^2\, L\right]$$

$$= \frac{\rho_0}{4} \pi R^2 L - \rho_0 \pi R^2 L + \frac{\rho_0}{4} \pi R^2 L = \frac{-\rho_0}{2} \pi R^2 L$$

Since this is negative, we know that the angle between $\vec{E}$ and $d\vec{A}$ in the determination of flux is actually 180°, not the 0° assumed. This makes the flux negative. So

$$\epsilon_0 \Phi = q_{\text{in}} \qquad \Rightarrow \qquad \epsilon_0\, (-E\, 4\pi\, RL) = \frac{-\rho_0}{2} \pi R^2 L \qquad \Rightarrow \qquad E = \frac{\rho_0 R}{8\epsilon_0}$$

2. (6 points) In the problem above, the charge per unit length of the hollow insulating cylinder is $\lambda_{\text{HC}} = -8\lambda_{\text{IC}}$, where λ_{IC} is the charge per unit length on the inner insulating cylinder. Consider two cylindrical Gaussian surfaces with the same length. Gaussian surface A has radius $R/2 < r_A < R$, and Gaussian surface B has radius $r_B = 4R$. What is the relationship between the electric fluxes Φ_A and Φ_B measured through the surfaces A and B, respectively?

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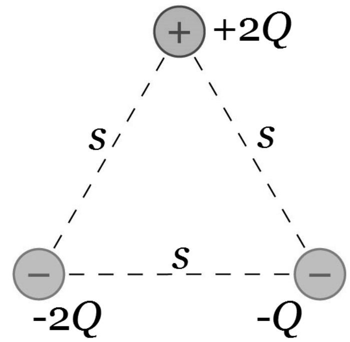
The flux through Gaussian Surface A is due to the charge on the inner cylinder, while the flux through Gaussian Surface B is due to the total charge. Since the charge in any given length of the hollow cylinder is -8 times the charge in that same length of inner cylinder, the total charge in that length must be -7 times the charge in the inner cylinder.

$$\Phi_B = -7\Phi_A$$

3. (8 points) Three particles with charges $+2Q$, $-2Q$, and $-Q$ are located at the vertices of an equilateral triangle with sides of length s. What is the electric potential energy of this system, with respect to zero at infinity?



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The electric potential energy of a system of point charges is the sum of the electric potentials of each pair of point charges considered individually. Letting the three point charges in this system be "top" T, "left" L, and "right" R, the electric potential energy of the system is

$$U_{\text{total}} = U_{TL} + U_{TR} + U_{LR} = K\frac{Q_T Q_L}{r_{\text{TL}}} + K\frac{Q_T Q_R}{r_{\text{TR}}} + K\frac{Q_L Q_R}{r_{\text{LR}}}$$

$$= K\frac{(+2Q)(-2Q)}{s} + K\frac{(+2Q)(-Q)}{s} + K\frac{(-2Q)(-Q)}{s} = \frac{K}{s}\left(-4Q^2 - 2Q^2 + 2Q^2\right) = -4KQ^2/s$$

4. (8 points) A negatively charged sphere is moved from some initial point to the final point in the direction of a uniform electric field. Considering a system consisting of the sphere and the source of the uniform field, during this displacement:

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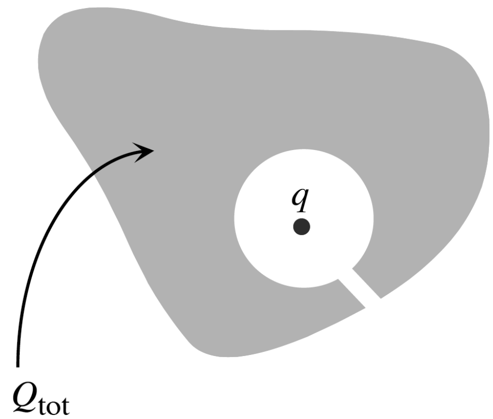
The negatively charged sphere is moved in the direction of the uniform electric field and therefore from some net positive toward a net negative distribution of charge. This kind of displacement requires an investment of additional energy, in order to move the sphere against the repulsive action of the electric force. Consequently,

the potential energy of the system increases.

5. (8 points) A conductor with a hollow cavity has a total charge Q_{tot} placed upon it. A small point charge q is then placed within the cavity, through a negligibly small hole in the conductor. What will be the resulting residual charge Q_{out} on the outer surface of the conductor, after equilibrium is reached?



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Use Gauss' Law. There is no flux through a Gaussian Surface drawn in the bulk of the conductor, as the field must be zero at equilibrium. Therefore, there must be no net charge inside that Gaussian Surface, which means the charge on the inner surface of the conductor must be opposite the point charge, $Q_{\text{in}} = -q$. Charge, however, is conserved, so

$$Q_{\text{tot}} = Q_{\text{out}} + Q_{\text{in}} = Q_{\text{out}} - q \qquad \Rightarrow \qquad Q_{\text{out}} = Q_{\text{tot}} + q$$

6. (8 points) Two parallel conducting plates are separated by 7.5 cm and one of them is taken to be at zero volts. What is the potential difference between the plates if the electric potential 5.0 cm from the zero volt plate (and 2.5 cm away from the other plate) is +300 Volts?

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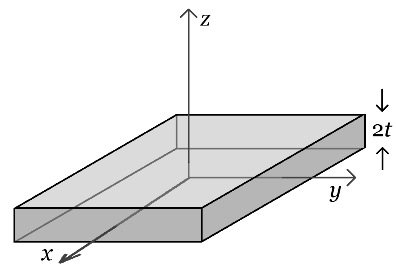
We are given the potential difference over two-thirds of the distance between the plates. This must be two-thirds of the potential difference over the full distance between the plates, as potential changes linearly between parallel plates. If two-thirds of the difference is +300 Volts, the entire difference must be

+450 Volts

7. (8 points) An infinite insulating slab has thickness $2t$. It extends to $\pm\infty$ in the x and y directions, and is centered on the z axis, extending to $\pm t$. It has a non-uniform volume charge density ρ that depends on position z according to

$$\rho = \rho_0 \left(\frac{z}{t}\right)^2$$

where ρ_0 is a constant. If it can be determined, what is the magnitude of the electric field at the origin?



Consider the symmetry of the charge distribution. For example, rotate the distribution 180° about the x axis. The charge distribution does not change with this transformation. Therefore, the field cannot change, either. The only way the direction of the field at the origin can remain unchanged when rotated 180° about the x axis is if it has no direction. The only way the field at a location can have no direction is if its magnitude is

Zero.