Physics 2212G/H Test form $\boxed{733}$ Name Solutions

Spring 2017

Test 3 Recitation Section (see back of test):________

A

1) Print your name, test form number (above), and nine-digit student number in the section of the answer card labeled "STUDENT IDENTIFICATION".

2) Bubble your test form number (**ABOVE**) in columns 1-3, skip column 4, then bubble in your student number in columns 5-13.

3) For each free-response question, show all relevant work supporting your answer. **Clearly box or underline your final answer.** "Correct" answers which are not supported by adequate calculations and/or reasoning will be counted wrong.

4) For each multiple-choice question, select the answer most nearly correct, **circle this answer on your test**, and bubble it in on your answer card. **Show all relevant work on your quiz.**

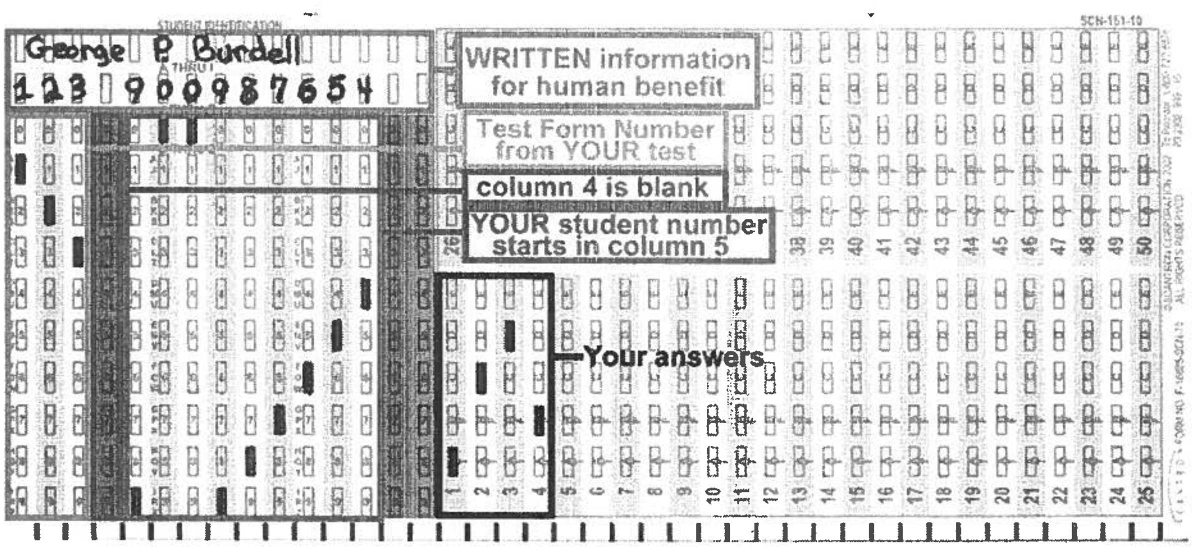
5) Be prepared to present your Buzzcard as you turn in your test. Scores will be posted to WebAssign after they have been been graded. **Quiz grades become final when the next quiz is given.**

6) You may use a simple scientific calculator capable of logarithms, exponentials, and trigonometric functions. **Programmable engineering calculators with text or graphical capabilities are not allowed. Wireless devices are prohibited.**

Numerical Constants:

$k = 8.99\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2$	$e = 1.60\times10^{-19}\ \text{C}$	$m_e = 9.11\times10^{-31}\ \text{kg}$
$\varepsilon_0 = 8.85\times10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2$	$g = 9.81\ \text{m/s}^2$	$m_p = 1.67\times10^{-27}\ \text{kg}$

Your test form is: $\boxed{733}$



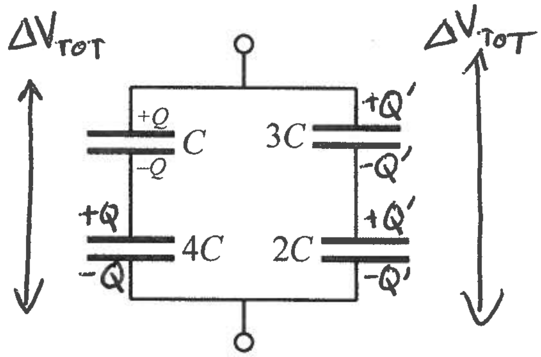
Our next test will be on Tuesday, April 11

The following problem will be hand-graded. <u>*Show all your work for this problem.*</u> *Make no marks and leave no space on your answer card for it.*

[I] *(20 points)* In the figure at right, a network of four capacitors have been hooked up to an emf (not shown), and charged up until eqilibrium has been established—after which the emf was removed and the network was isolated. A careful measurement of capacitor C reveals that it has a total charge Q stored on it.

Determine the charge stored on each of the other three capacitors, expressing each as a fraction or multiple of Q.

Determine the potential across each capacitor, expressed in terms of the parameters Q and C.



① Capacitor C has known charge and capacitance

→ clearly $\boxed{\Delta V_C = \frac{Q}{C}}$

② Capacitor 4C is in series with C

→ 4C must store the same charge as C $\boxed{Q_{4C} = Q}$

and hence, $\boxed{\Delta V_{4C} = \frac{Q}{4C}}$

③ Total potential across C and 4C, combined, is:

$$\Delta V_{TOT} = \Delta V_C + \Delta V_{4C} = \frac{Q}{C} + \frac{Q}{4C} = \frac{4Q}{4C} + \frac{Q}{C} = \frac{5Q}{4C}$$

④ combination 1+4 is in parallel with 2+3, so they must have the same ΔV_{TOT}

$$\Delta V_{2C} + \Delta V_{3C} = \frac{5Q}{4C}$$

⑤ 2C and 3C are in series, so they must store the same charge

→ call this unknown charge Q'

→ then this becomes $\frac{Q'}{2C} + \frac{Q'}{3C} = \frac{5Q}{4C}$

$$\frac{3Q'}{6C} + \frac{2Q'}{6C} = \frac{5Q}{4C} \rightarrow \frac{5Q'}{6C} = \frac{5Q}{4C} \rightarrow Q' = \frac{3}{2}Q$$

So, finally: $\boxed{Q_{2C} = Q_{3C} = \frac{3}{2}Q}$

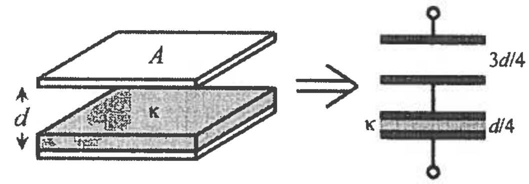
then $\Delta V_{2C} = \frac{3/2\,Q}{2C}$ and $\Delta V_{3C} = \frac{3/2\,Q}{3C}$

$\boxed{\Delta V_{2C} = \frac{3Q}{4C}}$ $\boxed{\Delta V_{3C} = \frac{Q}{2C}}$

Note that these two values add up to $5Q/4C$, as required!!

The following problem will be hand-graded. *Show all your work for this problem.* *Make no marks and leave no space on your answer card for it.*

[II] *(20 points)* A parallel-plate capacitor having area A and separation d has a capacitance in vacuum (or air) of $C_o = \varepsilon_o A/d$. You wish to make a capacitor filled with unicornium (dielectric constant $\kappa = 33$), but unicornium is rare, and you only have enough to fill your capacitor with a thickness $d/4$ of the material—the remainder of the gap between the plates will be filled with air ($\kappa = 1$). However, a *partially* filled capacitor can be modeled as an air-filled capacitor (plate separation $3d/4$) that is in series with a dielectric-filled capacitor (plate separation $d/4$).



(*i*) What will be the overall capacitance of the partially-filled capacitor? Express your answer as a numerical multiple of the vacuum/air capacitance value, C_o.

(*ii*) Charge $\pm Q$ is placed on the capacitor plates, resulting in surface charge density of magnitude $\eta_{cap} = Q/\cancel{\kappa}A$ on the conducting plates. What will be the magnitude of the polarization charge density η_{pol} induced on the surfaces of the unicornium? Express your answer as a numerical multiple of η_{cap}..

- For air gap, $C_{air} = \frac{\varepsilon_0 A}{3/4\, d} = \frac{4\varepsilon_0 A}{3d}$
- For unicornium layer $C_{un} = \kappa \frac{\varepsilon_0 A}{d/4} = \kappa \frac{4\varepsilon_0 A}{d}$

⇒ equivalent capacitance in series satisfies $C_{eq}^{-1} = \frac{1}{C_{air}} + \frac{1}{C_{un}}$

$$C_{eq}^{-1} = \frac{3d}{4\varepsilon_0 A} + \frac{d}{4\varepsilon_0 A\kappa} = \frac{d}{\varepsilon_0 A}\left[\frac{3}{4} + \frac{1}{4\kappa}\right]$$

$$C_{eq} = \frac{\varepsilon_0 A}{d}\left[\frac{3}{4} + \frac{1}{4\kappa}\right]^{-1} = \frac{\varepsilon_0 A}{d}\left[\frac{3}{4} + \frac{1}{132}\right]^{-1}$$

$$= \frac{\varepsilon_0 A}{d}\left[\frac{99}{132} + \frac{1}{132}\right]^{-1}$$

$$\boxed{C_{eq} = 1.32\frac{\varepsilon_0 A}{d} = 1.32 C_o}$$

- Field of a charged sheet is $E_0 = \frac{\eta}{\varepsilon_0} = \frac{Q/A}{\varepsilon_0}$
- inside dielectric, net field is $E_{in} = \frac{E_0}{\kappa}$ → this is field by charge on plates, partially canceled by field due to induced charge

ie $E_{in} = +E_0 - E_{pol}$

$$\rightarrow E_{pol} = E_0 - E_{in} = E_0 - \frac{E_0}{\kappa} = E_0\left(1 - \frac{1}{33}\right) = \frac{32}{33}\left[E_0\right]$$

↳ due to polarization charge, $\frac{\eta_{pol}}{\varepsilon_0}$ $\quad \frac{32}{33}\left[\frac{\eta_{cap}}{\varepsilon_0}\right]$

compare

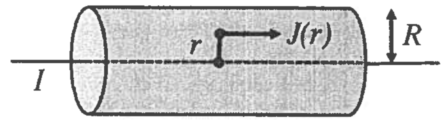
so $\boxed{\eta_{pol} = \frac{32}{33}\eta_{cap}}$

The following problem will be hand-graded. *<u>Show all your work for this problem</u>.* *Make no marks and leave no space on your answer card for it.*

[III] *(20 points)* A cylindrical conductor of radius R carries a non-uniform current density that depend on the distance r from the center of the wire:

$$J(r) = J_o \cdot [1 - r^2/R^2]$$

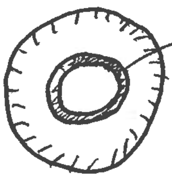
Here, J_o is a positively-valued constant.



(*i*) Determine the total current flowing in the wire. Express your answer in terms of R and J_o

(*ii*) What <u>fraction</u> of the total current is flowing in the <u>central</u> portion of the wire, extending from $r = 0$ to $r = R/2$?

Since J depends on distance from axis, we must sum over <u>thin</u> <u>rings</u> of cross-sectional area



J = constant on this ring of area $dA = 2\pi r \cdot dr$ [ie circumference x thickness]

$$dI = J(r)\,dA = J_o\left[1 - \frac{r^2}{R^2}\right]2\pi r\,dr$$

Hence, $I_{TOT} = \int_{r=0}^{r=R} dI = 2\pi J_o \int_0^R \left(r - \frac{r^3}{R^2}\right)dr$

$$= 2\pi J_o\left[\frac{r^2}{2} - \frac{r^4}{4R^2}\right]_0^R = 2\pi J_o\left[\frac{R^2}{2} - \frac{R^2}{4}\right]$$

so $\boxed{I_{TOT} = \frac{\pi}{2}R^2 J_o}$

To find fraction of I_{TOT} between 0 and $R/2$, follow <u>same</u> procedure but use limits $r=0$ to $r=R/2$

$$I_{frac} = 2\pi J_o\left[\frac{r^2}{2} - \frac{r^4}{4R^2}\right]_0^{R/2} = 2\pi J_o\left[\frac{R^2}{8} - \frac{R^2}{64}\right] = 2\pi J_o\left[\frac{7R^2}{64}\right]$$

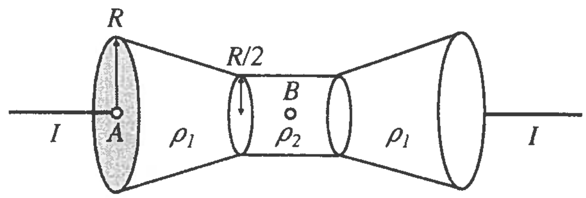
$$I_{frac} = \frac{7\pi}{32}R^2 J_o$$

$$\frac{I_{frac}}{I_{TOT}} = \frac{7/32}{1/2} = \boxed{\frac{7}{16}}$$

Just under <u>half</u> of all current flows through middle part of wire [even though the "middle part" is only <u>one fourth</u> of the cross-sectional area]

Question value 8 points

(1) A joint in a conducting wire is constructed from three pieces of material, as shown at right. The two flaring outer sections are made of iron, with resistivity ρ_1. Their radii vary smoothly from R at the ends to $R/2$ near the center. The narrow cylindrical neck is made of aluminum, with resistivity $\rho_2 = \rho_1/3$. A current I flows through the conductor. What is the ratio of the electric field magnitudes at point A (far right) and point B (center of the neck)?



(a) $E_B = 2/3\, E_A$
(b) $E_B = E_A$
(c) $E_B = 3/4\, E_A$
$\boxed{\text{(d) } E_B = 4/3\, E_A}$
(e) $E_B = 3/2\, E_A$

• same current I through all parts of wire

$I_A = I_B$

• noting that $I = JA$ for a <u>uniform</u> current distribution:

$J_A A_A = J_B A_B \rightarrow J_A \pi R^2 = J_B \pi (R/2)^2$

$\rightarrow J_A = \frac{1}{4} J_B$ or $\boxed{J_B = 4 J_A}$

• Ohm's Law: $J = \sigma E = \frac{1}{\rho} E \rightarrow E = \rho J$

so $E_A = \rho_1 J_A$

$E_B = \rho_2 J_B = \left(\frac{\rho_1}{3}\right)(4 J_A) = \frac{4}{3}(\rho_1 J_A) \rightarrow \boxed{E_B = \frac{4}{3} E_A}$

Question value 8 points

(2) A wire of cross-section A carries a time-dependent current, given by $I(t) = at^2$, where $a = 6.0\times10^{-3}$ A/s^2. How much total charge Q passed through A between $t = 0$ and $t = 10$ s?

(a) $Q = 1.8$ C
$\boxed{\text{(b) } Q = 2.0 \text{ C}}$
(c) $Q = 6.0$ C
(d) $Q = 1.2$ C
(e) $Q = 3.0$ C

$I = \frac{dQ}{dt}$ or $Q = \int I\,dt$

so $Q = \int_0^{t_1} (at^2)\,dt = a \int_0^{t_1} t^2\,dt$

$= a\left[\frac{t^3}{3}\right]_0^{t_1}$

$\boxed{Q = \frac{1}{3} a t_1^3}$

$= \frac{1}{3}(6\times10^{-3}\text{ A/s}^2)(10\text{s})^3$

$= 2.0 \times 10^0$ A·s

<u>$Q = 2.0$C</u>

Question value 8 points

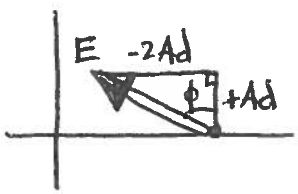
(3) The electric potential in the xy-plane is given by the expression $V(x,y) = A(x^2 - xy + y^2)$, where A is a positive constant, and x and y can take both positive and negative values. What is the <u>direction</u> of the electric field on the positive x axis, at a distance d from the origin (that is, as position $x = d$, $y = 0$)? Express your answer as an angle measured counterclockwise from the positive x-direction.

(a) $\theta = 63.4°$
(b) $\theta = 333°$
(c) $\theta = 180°$
$\boxed{\text{(d) } \theta = 153°}$
(e) $\theta = 315°$

$$E_x = \frac{-dV}{dx}\bigg|_{y \text{ held constant}} = -\left[A(2x-y)\right]$$

$$E_y = \frac{-dV}{dy}\bigg|_{x \text{ held constant}} = -\left[A(-x+2y)\right]$$

at $(x,y) = (d,0)$, this becomes: $E_x = -2Ad$, $E_y = +Ad$



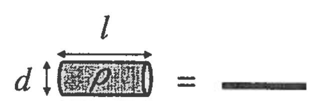
trig tells us: $\tan\phi = \frac{|E_x|}{|E_y|} = 2$ $\phi = \tan^{-1}(2) = 63.4°$

hence, angle from $+x$ axis to $\vec{E}$ is

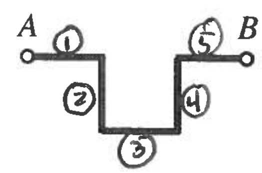
$$\boxed{\theta = \phi + 90° = 153°}$$

Question value 8 points

(4) Five identical cylindrical copper wires, having resistivity ρ, length l, and diameter d, are soldered together as shown at bottom right. What is the total resistance R of the resulting conductor, as measured between points A and B? *Hint:* Start by finding the resistance of just one segment of copper wire.



(a) $R = 3\rho\pi d^2/l$
(b) $R = 5\rho l/\pi d^2$
(c) $R = 5\rho\pi d^2 l$
(d) $R = \rho\pi d^2/3l$
$\boxed{\text{(e) } R = 20\rho l/\pi d^2}$



for a <u>single</u> segment,

$$R = \frac{\rho L}{A} = \frac{\rho L}{\pi(d/2)^2} = \frac{4\rho L}{\pi d^2}$$

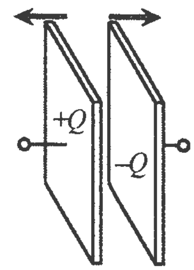
For <u>all</u> five segments, <u>total</u> length is $L_{TOT} = 5L$

hence, total resistance can be found by <u>neglecting</u> the 90° bends

$$R_{TOT} = \frac{\rho L_{TOT}}{A} = 5\frac{\rho L}{A} = \boxed{\frac{20\rho L}{\pi d^2}}$$

Question value 4 points

(5) An isolated parallel-plate capacitor has charge Q. The capacitor's plates are initially separated by a distance d. <u>While still isolated</u>, the plates are carefully pulled apart, increasing the separation to $2d$. If the energy initially stored by the capacitor was U_o, what will be the energy stored after increasing the plate separation?



(a) $U_f = U_o$
(b) $U_f = U_o/4$
(c) $U_f = 4U_o$
(d) $\boxed{U_f = 2U_o}$
(e) $U_f = U_o/2$

Energy in a capacitor

$U_c = \frac{1}{2}C(\Delta V)^2$ or $\frac{Q^2}{2C}$ or $\frac{1}{2}Q\Delta V$

since Q is <u>fixed</u> while C changes, 2nd formula suits us best

$U = Q^2/2C$

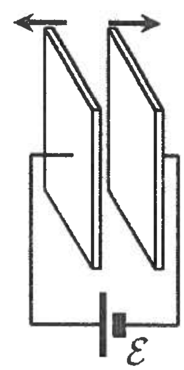
recall $C = \varepsilon_0 A/d$ so $d \to 2d$ means $C \to C/2$

$U_o = \frac{Q^2}{2C}$ $\quad U_f = \frac{Q^2}{2(C/2)} = \frac{2Q^2}{2C} = 2\left(\frac{Q^2}{2C}\right)$

$$\boxed{U_f = 2U_o}$$

Question value 4 points

(6) A parallel-plate capacitor is charged by attaching it to an emf $\mathcal{E}$. The capacitor's plates are initially separated by a distance d. <u>While still connected to the emf</u>, the plates are carefully pulled apart, increasing the separation to $2d$. If the energy initially stored by the capacitor was U_o, what will be the energy stored after increasing the plate separation?



(a) $U_f = 4U_o$
(b) $U_f = U_o$
(c) $U_f = U_o/4$
(d) $U_f = 2U_o$
(e) $\boxed{U_f = U_o/2}$

$U_c = \frac{1}{2}C(\Delta V)^2 = \frac{Q^2}{2C} = \frac{1}{2}Q\Delta V$

since ΔV is held fixed at $\mathcal{E}$, while C changes, 1st formula suits us best

$U_o = \frac{1}{2}C\mathcal{E}^2$, while $C \to C/2$

$U_f = \frac{1}{2}\left(\frac{C}{2}\right)\mathcal{E}^2 = \frac{1}{2}\left(\frac{1}{2}C\mathcal{E}^2\right)$

$$\boxed{U_f = \frac{1}{2}U_o}$$