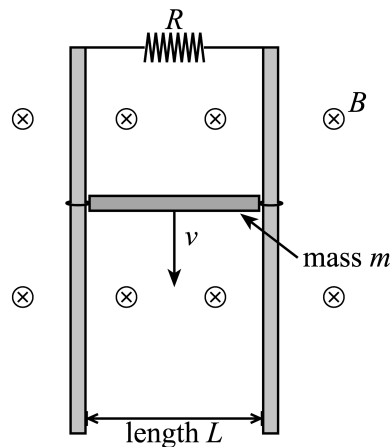


- I. (18 points) A bar (mass m , length L) is connected to two frictionless vertical conducting rails with loops of wire, in the presence of a uniform magnetic field $\vec{B}$. The tops of the rails are connected through a resistor R . When released from rest, the bar slides down the rails, accelerating more and more slowly until it reaches a constant terminal speed v . Find an expression for v in terms of parameters defined in the problem and physical or mathematical constants. (*On Earth. Do NOT neglect gravity.*)



When the constant terminal speed v is reached, Newton's Second Law tells us there must be no net force on the bar. The magnitude of the magnetic force on the bar must be equal to the magnitude of the gravitational force.

$$F_g = F_B \quad \Rightarrow \quad mg = ILB \sin \phi = ILB$$

as the angle between the direction of current flow (in the plane of the page) and the direction of the magnetic field (into the page) is 90° .

The current, then, is

$$I = \frac{mg}{LB}$$

Now consider the time rate at which the resistor converts electrical energy to thermal energy. This is the power dissipated in the resistor

$$P_R = I \Delta V = I^2 R$$

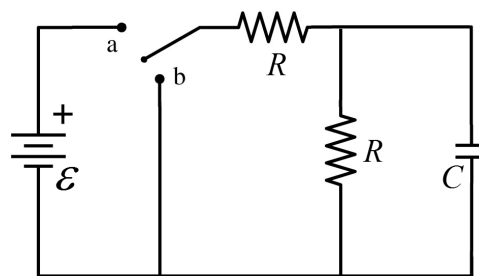
At the constant terminal speed, when the kinetic energy of the bar is constant, this must match the rate at which gravitational potential energy is being converted to electrical energy.

$$P_g = \frac{dU_g}{dt} = \frac{d}{dt}(mgh) = mg \frac{dh}{dt} = mgv$$

So

$$P_g = P_R \quad \Rightarrow \quad mgv = I^2 R \quad \Rightarrow \quad mgv = \left(\frac{mg}{LB}\right)^2 R \quad \Rightarrow \quad v = \frac{mgR}{L^2 B^2}$$

- II. (16 points) In the circuit shown, $\mathcal{E} = 9.0 \text{ V}$, $R = 100.0 \Omega$ and $C = 20.0 \mu\text{F}$. The switch has been in position “a” for a long time, then is thrown to position “b”. What is the current from the capacitor 1.8 ms after the switch is thrown?



After the switch has been at “a” for a long time, the capacitor is at full charge, so no current flows in the branch of the circuit containing the capacitor. Current flows from the emf through the two resistors, which have the same current and so are in series. That current is

$$I_a = \frac{\Delta V_R}{R_a} = \frac{\mathcal{E}}{R + R} = \frac{\mathcal{E}}{2R}$$

The potential across the capacitor must be the same as the potential across the vertical resistor. That potential is

$$\Delta V_{Ca} = \Delta V_R = I_a R = \left(\frac{\mathcal{E}}{2R} \right) R = \frac{\mathcal{E}}{2}$$

As the switch is thrown from “a” to “b”, the charge on the capacitor, and therefore the potential across the capacitor, remains constant. The initial potential across the capacitor when the switch arrives at “b” is

$$\Delta V_{C0b} = \Delta V_{Ca} = \frac{\mathcal{E}}{2}$$

Note that this is also the potential across each resistor. The resistors are now, therefore, in parallel.

$$R_{\text{eq}} = \left(\frac{1}{R} + \frac{1}{R} \right)^{-1} = \frac{R}{2}$$

so the initial current from the capacitor is

$$I_0 = \frac{\Delta V_{C0b}}{R_{\text{eq}}} = \frac{\mathcal{E}/2}{R/2} = \frac{\mathcal{E}}{R}$$

This current decays with time.

$$\begin{aligned} I &= I_0 e^{-t/\tau} = \frac{\mathcal{E}}{R} e^{-t/R_{\text{eq}}C} = \frac{\mathcal{E}}{R} e^{-t/(R/2)C} = \frac{\mathcal{E}}{R} e^{-2t/RC} \\ &= \frac{9.0 \text{ V}}{100.0 \Omega} e^{-2(1.8 \times 10^{-3} \text{ s})/(100.0 \Omega)(20.0 \times 10^{-6} \text{ F})} = 1.5 \times 10^{-2} \text{ A} = \mathbf{15 \text{ mA}} \end{aligned}$$

1. (5 points) At the instant the switch is thrown to “b” in the problem above, let the current from the capacitor be I_0 and the potential energy stored in the capacitor be U_0 , with respect to zero in the uncharged state. When the current from the capacitor has decreased to half its initial value ($I = I_0/2$), what potential energy is stored in the capacitor?

Since the resistances don’t change, when the current has reached half its initial value, the potential across them, and therefore the potential across the capacitor, must be half its initial value. The potential energy stored in a capacitor is

$$U = \frac{1}{2} C (\Delta V)^2$$

so when the potential has dropped to half its initial value, the stored energy must be one-fourth of its initial value.

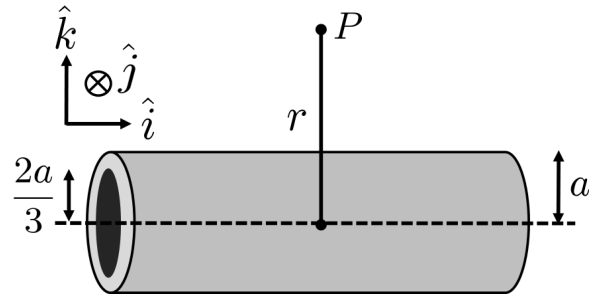
$$\mathbf{U = U_0/4}$$

- III. (16 points) An infinitely long hollow wire has inner radius $2a/3$ and outer radius a . A uniform current density $\vec{J}_0$ flows in the $\hat{i}$ direction in the conductor. No current flows in the hollow. What is the magnitude of the magnetic field created at a point P located at a distance $r > a$ from the axis of the wire? Express your answer in terms of parameters defined in the problem and physical or mathematical constants.

Use Ampere's Law

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{\text{thru}}$$

Choose a path that passes through the point P that will make $\oint \vec{B} \cdot d\vec{s}$ easy to evaluate. This path will follow a field line, making the magnitude of $\vec{B}$ constant, and everywhere parallel to $d\vec{s}$. Such a path is a circle of radius r .



$$\oint \vec{B} \cdot d\vec{s} = \oint B \cos \theta ds = B 2\pi r$$

The current through the surface defined by this path is the total current in the wire, which can be related to the current density. Choosing the area vector to point in the same direction as the current density,

$$I_{\text{thru}} = \int \vec{J} \cdot d\vec{A} = J_0 A$$

as the current density J_0 is constant. The area through which current flows is the area of a circle with radius a , minus the area of the hollow.

$$I_{\text{thru}} = J_0 \left[\pi a^2 - \pi \left(\frac{2}{3}a \right)^2 \right] = \pi J_0 \left[a^2 - \frac{4}{9}a^2 \right] = \frac{5}{9} \pi J_0 a^2$$

So

$$B 2\pi r = \mu_0 \frac{5}{9} \pi J_0 a^2 \quad \Rightarrow \quad B = \frac{5\mu_0 J_0 a^2}{18r}$$

2. (5 points) Assuming $J_0 > 0$ in the problem above, what is the direction of the magnetic field at the point P ?

Use the Right-Hand-Rule with Ampere's Law. Point the thumb of your right hand in the direction of current flow (to the right, the $+\hat{i}$ direction), then curl your fingers through the point P . They'll be pointing out of the page, which is the direction

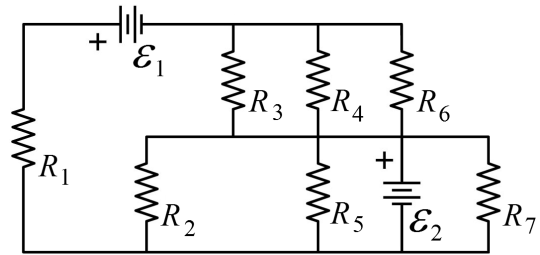
$$-\hat{j}$$

3. (5 points) Let the positive direction of current flow be up the page in the circuit shown. The current I_1 flows through resistor R_1 , etc. Which is a valid expression of Kirchhoff's Loop Law?

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With the positive current direction defined as up the page through each resistor, the electric potential increases when traveling down the page through a resistor, and increases when traveling up the page. The only expression of Kirchhoff's Loop Law offered that both represents a closed path and has all these potential changes correct is

$$+\mathcal{E}_1 + I_1 R_1 + \mathcal{E}_2 - I_4 R_4 = 0$$

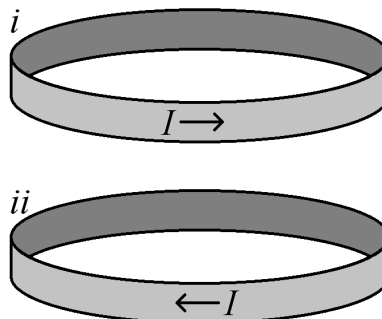


4. (5 points) Identical conducting rings, i and ii , carry currents I of identical magnitude, but that travel in opposite senses, as shown. What is the direction of the magnetic force, if any, on each ring?

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Using the Right-Hand-Rule shortcut from the Biot-Savart Law, run the fingers of your right hand around loop i . Your thumb points in the direction of the magnetic field in the center of the loop. This is up the page. As magnetic field lines come out the North end of a magnetic dipole, the North end of loop i 's dipole is above it, and the South end is below it.

Doing the same for loop ii , you'll find that the the North end of loop ii 's dipole is below it, and the North end is below it.



These dipoles, then, both have South poles between the loops. Like poles repel, so

The force on ring i is up the page, but the force on ring ii is down the page.

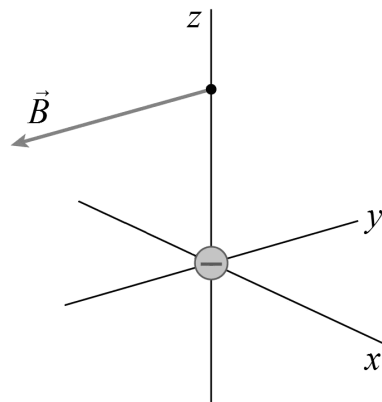
5. (5 points) As a particle with a charge of -1.0 nC passes through the origin, it generates a magnetic field on the $+z$ axis that points in the $-y$ direction, as shown. In what direction could this negatively-charged particle be traveling?

Use the Biot-Savart Law.

$$\vec{B} = \frac{\mu_0 q}{4\pi} \frac{\vec{v} \times \hat{r}}{|\vec{r}|^2}$$

Since the particle has a negative charge, the magnetic field must be opposite the direction of $\vec{v} \times \hat{r}$. The direction opposite the magnetic field is $+\hat{y}$. The direction of $\hat{r}$ is $+\hat{z}$. As $-\hat{x} \times \hat{z} = \hat{y}$, the particle could be traveling

In the $-x$ direction.



6. (5 points) In the problem above, the magnetic field has a magnitude of 2.5 nT at a position $z = +9.8\text{ cm}$ as the particle passes through the origin. What is the minimum speed of the particle?

Use the Biot-Savart Law.

$$\vec{B} = \frac{\mu_0 q}{4\pi} \frac{\vec{v} \times \hat{r}}{|\vec{r}|^2} \quad \Rightarrow \quad B = \frac{\mu_0 |q|}{4\pi} \frac{v \sin \phi}{r^2}$$

Note that the minimum speed is achieved when $\sin \phi$ has its maximum value of “1”. The “nanos” cancel.

$$B = \frac{\mu_0 |q| v}{4\pi r^2} \quad \Rightarrow \quad v = \frac{B 4\pi r^2}{\mu_0 |q|} = \frac{(2.5\text{ nT}) 4\pi (9.8 \times 10^{-2}\text{ m})^2}{4\pi \times 10^{-7}\text{ T}\cdot\text{m/A} (1.0\text{ nC})} = \frac{(2.5\text{ T}) (9.8 \times 10^{-2}\text{ m})^2}{10^{-7}\text{ T}\cdot\text{m/A} (1.0\text{ C})} = 2.4 \times 10^5\text{ m/s}$$

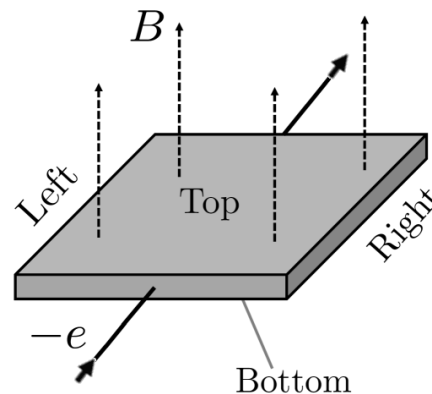
7. (5 points) Electrons are injected in a sheet of conducting material immersed in a uniform magnetic field B as shown. Electrons are deflected ...

The magnetic force on the moving electrons is

$$\vec{F} = q\vec{v} \times \vec{B}$$

which is to the left (remember that electrons are negative, so the force is opposite the direction of $\vec{v} \times \vec{B}$). The sheet remains electrically neutral, so positive charge remains on the right. Electric field points from positive to negative charge (that is, from right to left), and also from high to low potential. Therefore, electrons are deflected ...

and a difference of potential is created between the left and right faces with $V_R - V_L > 0$.



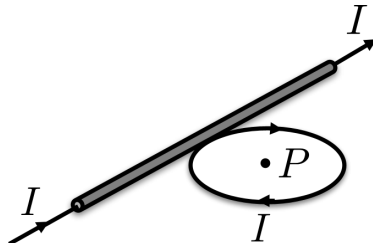
8. (5 points) A infinitely long thin wire and a current loop of radius R lie in the same plane and are touching as shown. A current I flows in the wire and in the loop in the directions indicated on the drawing. What is the magnitude of the magnetic field at point P ?

Use Ampere's Law to find the field due to the long wire.

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{\text{thru}}$$

$$B 2\pi R = \mu_0 I$$

$$B_{\text{long}} = \frac{\mu_0 I}{2\pi R}$$



down the page, from the Right-Hand-Rule.

Use the Biot-Savart Law to find the field due to the loop of wire.

$$\vec{B} = \int d\vec{B} = \int \frac{\mu_0 I}{4\pi} \frac{d\vec{\ell} \times \hat{r}}{|\vec{r}|^2} \Rightarrow B_{\text{loop}} = \frac{\mu_0 I}{4\pi} \int_0^{2\pi R} \frac{d\ell \sin \frac{\pi}{2}}{R^2} = \frac{\mu_0 I}{4\pi R^2} \int_0^{2\pi R} d\ell = \frac{\mu_0 I}{4\pi R^2} 2\pi R = \frac{\mu_0 I}{2R}$$

also down the page, from the Right-Hand-Rule. Since the fields due to the long straight wire and the loop both are directed down the page, the magnitude of the field at the center of the loop is the sum of the individual fields.

$$B_{\text{center}} = B_{\text{long}} + B_{\text{loop}} = \frac{\mu_0 I}{2\pi R} + \frac{\mu_0 I}{2R} = \frac{\mu_0 I}{2R} \left(\frac{1}{\pi} + 1 \right) = \frac{\mu_0 I}{2R} \left(\frac{1}{\pi} + 1 \right) = \frac{\mu_0 I}{2R} \frac{1 + \pi}{\pi}$$

9. (5 points) The accompanying figure shows the tracks of several charged particles in the presence of magnetic fields. The magnetic field is uniform and into the page on the left; it is uniform and out of the page on the right. Which of these particles is positively charged, and which negatively?

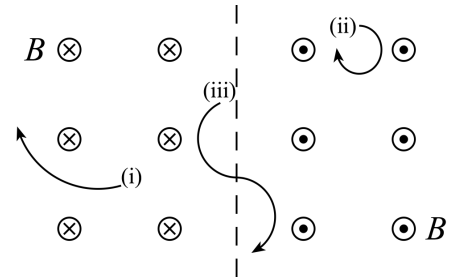
The force on a moving charge in a magnetic field is

$$\vec{F} = q\vec{v} \times \vec{B}$$

so the force on a positive charge will be in the direction of $\vec{v} \times \vec{B}$, while the force on a negative charge will be in the opposite direction. The path of the charge will be deflected in the direction of the force.

Using the Right-Hand-Rule for the vector product, $\vec{v} \times \vec{B}$ is opposite the path deflection for particle *i*. It is in the direction of the path deflection for particle *ii*. It is in the direction of the path deflection for each portion of the path of particle *iii*. Therefore,

(i) is negative; *(ii)* is positive; *(iii)* is positive.



10. (5 points) A long wire of length L and width (or diameter) w is to be used to make a solenoid. The entire wire will be wrapped in a single layer, with adjacent coils touching (that is, packed as closely together as possible without overlapping). A known, fixed current I will flow through the wire, making a magnetic field inside the solenoid. You may assume that the resulting solenoid is much longer than its diameter. What is the magnitude of the magnetic field inside the solenoid?

Since this solenoid is much longer than its diameter, the magnetic field within it can be approximated by that of an ideal infinite solenoid.

$$B = \mu_0 I n = \mu_0 I \frac{N}{\ell}$$

where N is the number of turns of wire making up the solenoid, and ℓ is the length of the solenoid.

Each turn of wire will be as wide as the wire, so the length of the solenoid is $\ell = Nw$. Substituting,

$$B = \mu_0 I n = \mu_0 I \frac{N}{Nw} \Rightarrow B = \mu_0 I / w$$