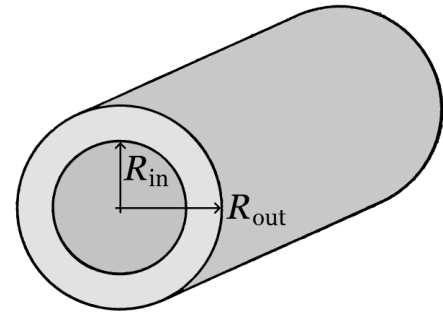


- I. (18 points) A hollow conducting wire has inner radius  $R_{\text{in}}$  and outer radius  $R_{\text{out}}$ . Between those radii, it has a non-uniform current density  $\vec{J}$ , whose magnitude depends on distance  $r$  from the central axis according to

$$J = J_0 \left( \frac{R_{\text{out}}}{r} \right)^2$$

where  $J_0$  is a positive constant. What is the current carried by the wire, in terms of parameters defined in the problem, and physical or mathematical constants?



Current can be related to current density:

$$I = \int \vec{J} \cdot d\vec{A}$$

Choose  $d\vec{A}$  to be “small” in the direction of the current density’s variation (the  $r$  direction). Such a  $d\vec{A}$  is a thin ring of radius  $r$  and width  $dr$ . Let its direction be parallel to  $\vec{J}$ .

$$I = \int_{R_{\text{in}}}^{R_{\text{out}}} J_0 \left( \frac{R_{\text{out}}}{r} \right)^2 \cos(0) 2\pi r dr = 2\pi J_0 R_{\text{out}}^2 \int_{R_{\text{in}}}^{R_{\text{out}}} \frac{dr}{r} = 2\pi J_0 R_{\text{out}}^2 \ln r \Big|_{R_{\text{in}}}^{R_{\text{out}}} = 2\pi J_0 R_{\text{out}}^2 \ln \left( \frac{R_{\text{out}}}{R_{\text{in}}} \right)$$

- II. (16 points) The network of capacitors shown is connected to a battery of emf  $\mathcal{E} = 24 \text{ V}$  where  $C = 6.0 \text{ nF}$ . Calculate the potential energy  $U_0$  stored in the network of capacitors once equilibrium is reached, relative to zero in the uncharged state.

Find the equivalent capacitance. The  $4C$  and  $3C$  capacitors on the left are in series.

$$C_{\text{eq left}} = \left( \frac{1}{4C} + \frac{1}{3C} \right)^{-1} = \frac{12}{7}C$$

The  $2C$  and  $C$  capacitors on the right are in parallel.

$$C_{\text{eq right}} = 2C + C = 3C$$

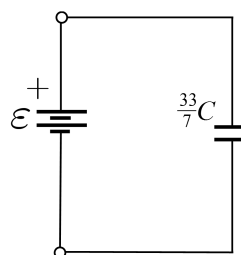
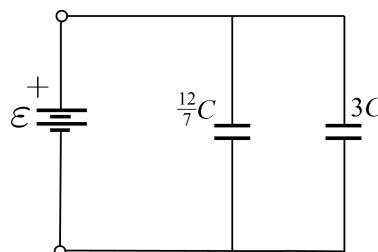
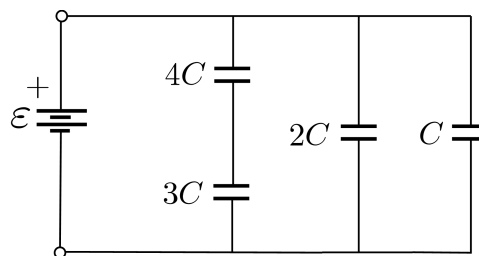
These two resulting capacitors are in parallel.

$$C_{\text{eq total}} = \frac{12}{7}C + 3C = \frac{33}{7}C$$

The potential across this equivalent capacitance is the emf of the battery, and if it is truly the “equivalent” capacitance, the energy stored in it must be the energy stored in the network.

$$U_0 = \frac{1}{2}C(\Delta V)^2 = \frac{1}{2} \left( \frac{33}{7} \times 6.0 \text{ nF} \right) (24 \text{ V})^2 = 8.1 \times 10^3 \text{ nJ}$$

$$= 8.1 \mu\text{J}$$



1. (5 points) The emf of the battery is increased to  $\mathcal{E} = 48 \text{ V}$ . How much potential energy  $U$  is now stored in the network of capacitors?

The energy stored in each capacitor is proportional to the square of the potential across that capacitor. But the potential across each capacitor must be proportional to the emf. Therefore, if the emf of the battery is doubled, the potential across each capacitor in the network is doubled, and the energy in the system increases by a factor of four.

$$U = 4U_0$$

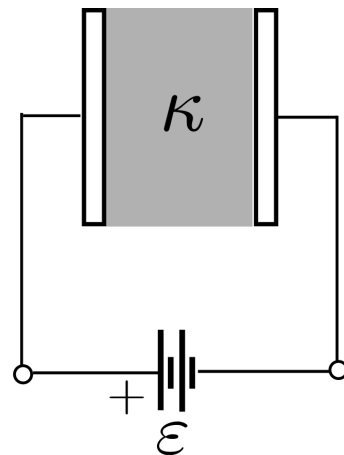
2. (5 points) A capacitor contains a dielectric material of dielectric constant  $\kappa = 3.4$ . The capacitance of the capacitor is  $C = 12\ \mu\text{F}$  when the dielectric material is inside. The capacitor is connected to a battery of emf  $\mathcal{E} = 10.0\ \text{V}$ . The battery remains connected to the capacitor and the dielectric is removed from the capacitor.

What, if anything, remains constant as the dielectric is removed?

.....

Since the battery remains connected to the capacitor, what remains constant is

The potential difference across the capacitor.



- III. (16 points) Calculate the work required for agents external to the capacitor to remove the dielectric at constant speed.

.....

Use the Work-energy Theorem, with the capacitor and dielectric as the system. The potential energy of the system is that stored in the capacitor. If the dielectric is removed at constant speed, there is no kinetic energy change. No non-conservative forces act within the system to change its thermal energy.

$$W_{\text{ext}} = \Delta K + \Delta U + \Delta E_{\text{th}} \quad \Rightarrow \quad W_{\text{ext}} = W_{\text{ext}} = 0 + \Delta U + 0$$

Since the battery remains connected to the capacitor, keeping the potential across it constant, it is simplest to use  $U = \frac{1}{2}C(\Delta V)^2$  for the potential energy stored in the capacitor.

$$W_{\text{ext}} = \frac{1}{2}C_f(\Delta V)^2 - \frac{1}{2}C_i(\Delta V)^2$$

The initial capacitance, with the dielectric in the capacitor, is given. When the dielectric is removed,  $C_f = C_i/\kappa$ , so

$$W_{\text{ext}} = \left( \frac{1}{2} \frac{C_i}{\kappa} (\Delta V)^2 - \frac{1}{2} C_i (\Delta V)^2 \right) = \frac{1}{2} C_i (\Delta V)^2 \left( \frac{1}{\kappa} - 1 \right) = \frac{1}{2} (12\ \mu\text{F}) (10.0\ \text{V})^2 \left( \frac{1}{3.4} - 1 \right) = -420\ \mu\text{J}$$

3. (5 points) The graph shows the electric potential  $V$  as a function of position  $x$ . In which region of this range will the most negative electric field be found (that is, the negative electric field with the greatest magnitude)?

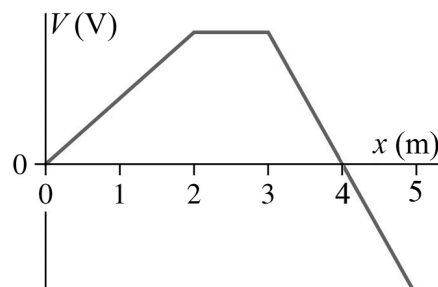
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The relationship between electric field and potential is

$$E_x = -\frac{\delta V}{\delta x}$$

for field in the  $x$  direction. So the field will only be negative in a region where the given graph has positive slope, which is

Somewhere between 0 and 2 meters.



4. (5 points) The graph shows the value of a one-dimensional electric field  $\vec{E}$  as a function of position  $x$ . At which point in this range is the electric potential greatest?

.....

The relationship between electric field and potential is

$$\Delta V = -\int \vec{E} \cdot d\vec{s}$$

so the potential *difference* between two points is the opposite of the area under the curve in the given graph.

The area under the curve between points  $i$  and  $ii$  is positive, so point  $ii$  is at a lower potential than point  $i$ . The potential cannot be greatest at point  $ii$ .

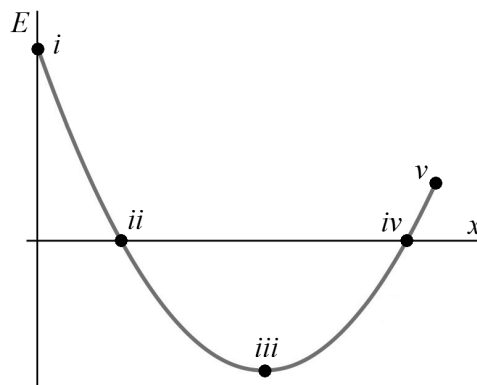
From point  $i$  to point  $iii$ , the net area under the curve is a bit negative. So point  $iii$  is at a higher potential than point  $i$ . The potential cannot be greatest at point  $i$ .

From point  $iii$  to point  $iv$ , the area under the curve is negative. Point  $iv$  is at an even higher potential than point  $iii$ . The potential cannot be greatest at point  $iii$ .

From point  $iv$  and  $v$ , the area under the curve is positive, so point  $v$  is at a lower potential than point  $iv$ . The potential cannot be greatest at point  $v$ .

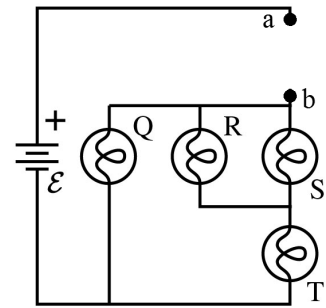
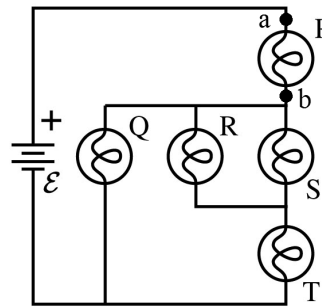
Therefore, the electric potential is greatest at

Point  $iv$ .



5. (5 points) When the five identical bulbs  $P$ ,  $Q$ ,  $R$ ,  $S$ , and  $T$  are attached to emf  $\mathcal{E}$  as shown on the left, all five light up. If bulb  $P$  is removed, as shown on the right, what is the potential difference **from** point “a” **to** point “b”?

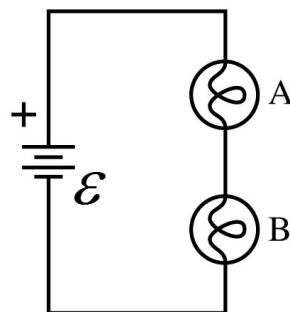
After bulb  $P$  is removed, there is no complete path between the high and low potential terminals of the battery. No current can flow, so there is no potential difference across any of the remaining bulbs. Applying Kirchhoff’s Loop Law, the potential across the gap must be the same as the potential across the battery. Since point “a” is connected to the high potential terminal of the battery, and point “b” is attached to the low potential terminal, potential decreases from “a” to “b”. The potential difference **from** point “a” **to** point “b”, then, is



$$-\mathcal{E}$$

6. (5 points) Bulbs “A” and “B” are ordinary 120 V household bulbs. Bulb “A” is rated at 150 W, and bulb “B” is rated at 60 W. They are connected to a 120 V emf  $\mathcal{E}$  as shown. Which bulb, if either, is brighter? Which bulb will be brighter if the position of the bulbs is swapped?

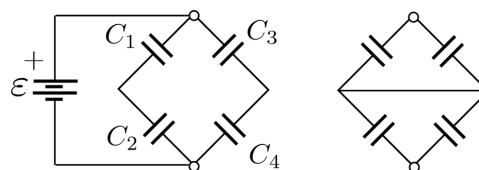
When the bulbs are attached individually across 120 V (normal household operation), bulb “A”, with the greater power rating,  $P = (\Delta V)^2 / R$  shows us that it must have a lower resistance than bulb “B”. However, when connected in series as shown,  $P = I^2 R$  shows us that the bulb with lower resistance transforms energy at a lower rate (i.e., is dimmer). Remember that bulbs in series have the same current, regardless of their order in the circuit.



Bulb “B” is brighter, and will still be brighter if the positions are swapped.

7. (5 points) The capacitor network below is connected to a battery and charged to a positive potential energy  $U_0$ . The network is then disconnected from the battery and a wire is connected as shown. How does the potential energy  $U$  of the network change after the wire is connected? *Note:* The capacitors are all different. Define the potential energy of a capacitor to be zero when it is uncharged.

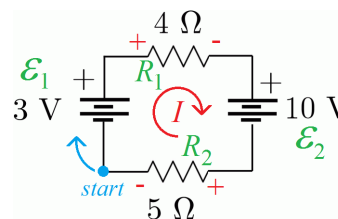
Since the capacitors are all different, then before the new wire is connected, the two “corners” of the circuit will be at different potentials. Once the wire is connected, charge must flow until the two corners are at the same potential. Since this is spontaneous, the system must be moving to a lower potential energy configuration. But since the potential energy stored in a capacitor is  $U = \frac{1}{2}C(\Delta V)^2 = \frac{1}{2}Q^2/C$ , it cannot be zero or negative.



The potential energy decreases, but remains positive.

8. (5 points) What is the magnitude and the direction of the current in the circuit?

Use Kirchhoff's Loop Law. Choose a current direction. I've chosen clockwise. Remembering that current flows from high to low potential through a resistor, mark the high and low potential sides of the resistors. Choose a starting point for applying the Loop Law, and a direction for traveling around the loop, adding up the potential changes. Note that the current must be the same through the two resistors.



$$+\mathcal{E}_1 - IR_1 - \mathcal{E}_2 - IR_2 = 0$$

Solve for the current  $I$ .

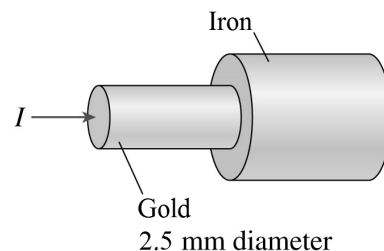
$$I = \frac{\mathcal{E}_1 - \mathcal{E}_2}{R_1 + R_2} = \frac{3\text{ V} - 10\text{ V}}{4\Omega + 5\Omega} = -0.78\text{ A}$$

The negative sign indicates that the chosen current direction, clockwise, is incorrect.

$I = 0.78\text{ A}$  counterclockwise.

9. (5 points) Current flows through a 2.5 mm diameter gold wire, then into an iron wire, as shown. Considering that the conductivity of gold is  $4.0 \times 10^7 \Omega^{-1}\text{m}^{-1}$  and that of iron is  $1.0 \times 10^7 \Omega^{-1}\text{m}^{-1}$ , what diameter must the iron wire have, if the electric field magnitude in the iron wire is to be the same as that in the gold wire? *Note:* Although the iron wire has been pictured greater in diameter than the gold wire, no conclusion should be drawn from that.

The currents in the two wires must be the same. They can be related to the current densities in the wires, which can, in turn, be related to the electric fields.



$$I_{\text{Au}} = I_{\text{Fe}} \Rightarrow J_{\text{Au}} A_{\text{Au}} = J_{\text{Fe}} A_{\text{Fe}} \Rightarrow \sigma_{\text{Au}} E_{\text{Au}} A_{\text{Au}} = \sigma_{\text{Fe}} E_{\text{Fe}} A_{\text{Fe}}$$

Remember the given condition that  $E_{\text{Au}} = E_{\text{Fe}}$ .

$$\sigma_{\text{Au}} A_{\text{Au}} = \sigma_{\text{Fe}} A_{\text{Fe}} \Rightarrow A_{\text{Fe}} = A_{\text{Au}} \frac{\sigma_{\text{Au}}}{\sigma_{\text{Fe}}} \Rightarrow \frac{\pi}{4} D_{\text{Fe}}^2 = \frac{\pi}{4} D_{\text{Au}}^2 \frac{\sigma_{\text{Au}}}{\sigma_{\text{Fe}}}$$

So

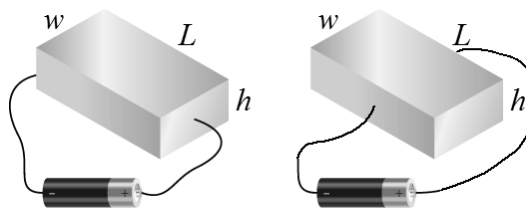
$$D_{\text{Fe}} = D_{\text{Au}} \sqrt{\frac{\sigma_{\text{Au}}}{\sigma_{\text{Fe}}}} = (2.5 \text{ mm}) \sqrt{\frac{4.0 \times 10^7 \Omega^{-1}\text{m}^{-1}}{1.0 \times 10^7 \Omega^{-1}\text{m}^{-1}}} = (2.5 \text{ mm}) \sqrt{4} = 2(2.5 \text{ mm}) = \mathbf{5.0 \text{ mm}}$$

10. (5 points) A uniform rectangular conducting block has width  $w = 3.0 \text{ cm}$ , length  $L = 6.0 \text{ cm}$ , and height  $h = 1.2 \text{ cm}$ . When a 4.5 V battery is connected across the ends (faces of area  $w \times h$ , as shown on the left), a 16 mA current flows. What current flows when the same battery is connected across the front and back (faces of area  $L \times h$ , as shown on the right)?

Use the definition of resistance and the relationship between resistance and resistivity.

$$\Delta V = IR \quad \text{and} \quad R = \rho \frac{\ell}{A}$$

Since the same battery is used, the potential is the same in both situations.



$$\Delta V = I_{\text{left}} R_{\text{left}} = I_{\text{right}} R_{\text{right}} \Rightarrow I_{\text{left}} \rho \frac{\ell_{\text{left}}}{A_{\text{left}}} = I_{\text{right}} \rho \frac{\ell_{\text{right}}}{A_{\text{right}}}$$

Since the block isn't changed, the resistivity is the same in both situations.

$$I_{\text{left}} \frac{\ell_{\text{left}}}{A_{\text{left}}} = I_{\text{right}} \frac{\ell_{\text{right}}}{A_{\text{right}}} \Rightarrow I_{\text{right}} = I_{\text{left}} \left( \frac{\ell_{\text{left}}}{\ell_{\text{right}}} \right) \left( \frac{A_{\text{right}}}{A_{\text{left}}} \right)$$

So

$$I_{\text{right}} = I_{\text{left}} \left( \frac{L}{w} \right) \left( \frac{Lh}{wh} \right) = I_{\text{left}} \left( \frac{L}{w} \right)^2 = (16 \text{ mA}) \left( \frac{6.0 \text{ cm}}{3.0 \text{ cm}} \right)^2 = 4(16 \text{ mA}) = \mathbf{64 \text{ mA}}$$