

I. (18 points) For the circuit shown, what power is dissipated by the $36\ \Omega$ resistor?

Finding the equivalent resistance of the circuit will enable the current through, or the potential across, the $36\ \Omega$ resistor to be determined. With that, the power can be calculated. The resistances have been assigned symbols, as shown on the figure.

Resistors R_1 and R_2 are in parallel.

$$R_{12} = \left(\frac{1}{R_1} + \frac{1}{R_2} \right)^{-1} = \left(\frac{1}{36\ \Omega} + \frac{1}{12\ \Omega} \right)^{-1} = \left(\frac{4}{36\ \Omega} \right)^{-1} = 9.0\ \Omega$$

Resistors R_{12} , R_3 , and R_4 are in series.

$$R_{1234} = R_{12} + R_3 + R_4 = 9.0\ \Omega + 4\ \Omega + 2\ \Omega = 15\ \Omega$$

From the definition of resistance, the current through resistor R_{1234} is

$$I_{1234} = \Delta V_{1234} / R_{1234} = \mathcal{E} / R_{1234} = \frac{75\ \text{V}}{15\ \Omega} = 5.0\ \text{A}$$

Since resistors R_{12} , R_3 , and R_4 are in series, this must also be the current through resistor R_{12} .

$$I_{12} = I_{1234} = 5.0\ \text{A}$$

From the definition of resistance, the potential across resistor R_{12} can be determined.

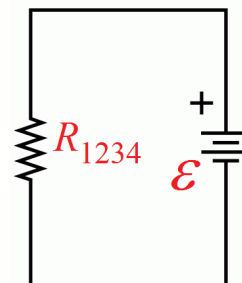
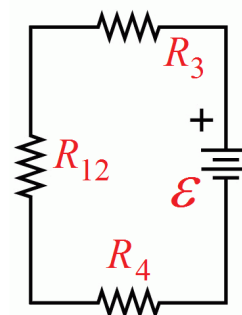
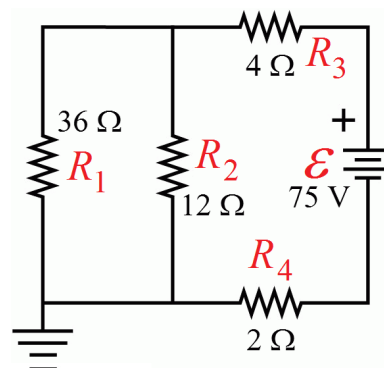
$$\Delta V_{12} = I_{12} R_{12} = (5.0\ \text{A})(9.0\ \Omega) = 45\ \text{V}$$

Since resistors R_1 and R_2 are in parallel, this must also be the potential across resistor R_1 .

$$\Delta V_1 = \Delta V_{12} = 45\ \text{V}$$

The power dissipated by a resistor is

$$P = I \Delta V = \left(\frac{\Delta V}{R} \right) \Delta V \quad \Rightarrow \quad P_1 = \frac{(\Delta V_1)^2}{R_1} = \frac{(45\ \text{V})^2}{36\ \Omega} = 56\ \text{W}$$

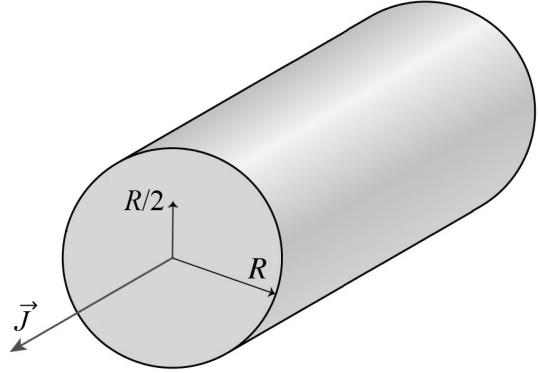


II. (16 points) A long, straight conducting wire of radius R has a nonuniform current density directed out of the page, with magnitude

$$J(r) = J_0 \left(\frac{r^2}{R^2} \right)$$

where r is the distance from the center of the wire and J_0 is a positive constant.

What is the magnitude of the magnetic field at a distance $R/2$ from the center? Express your answer in terms of parameters defined in the problem, and physical or mathematical constants.



Use Ampere's Law, $\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{\text{thru}}$. Choose a circular Amperian Loop concentric with the wire, and that passes through the point at which the field is to be determined (that is, with radius $R/2$). With that choice, the field magnitude is uniform on the Loop, and the field direction is everywhere tangent to the loop. So

$$\oint \vec{B} \cdot d\vec{s} = B2\pi r = B2\pi \left(\frac{R}{2} \right) = B\pi R$$

The current through the Loop can be determined from the current density, as they are related by $I = \int \vec{J} \cdot d\vec{A}$. Since the current density varies in with radius, choose area elements dA that are small in the “ r ” direction. These are thin rings, $dA = 2\pi r dr$. With $d\vec{A}$ parallel to \vec{J} ,

$$I = \int \vec{J} \cdot d\vec{A} = \int_0^{R/2} J_0 \left(\frac{r^2}{R^2} \right) 2\pi r dr = \frac{2\pi J_0}{R^2} \int_0^{R/2} r^3 dr = \frac{2\pi J_0}{R^2} \left[\frac{r^4}{4} \right]_0^{R/2} = \frac{2\pi J_0}{4R^2} \left(\frac{R}{2} \right)^4 = \frac{\pi J_0 R^2}{32}$$

Finally,

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{\text{thru}} \quad \Rightarrow \quad B\pi R = \mu_0 \frac{\pi J_0 R^2}{32} \quad \Rightarrow \quad B = \mu_0 \frac{J_0 R}{32}$$

1. (5 points) In the problem above, what is the direction of the magnetic field at the point a distance $R/2$ above the center of the wire, as indicated on the figure?

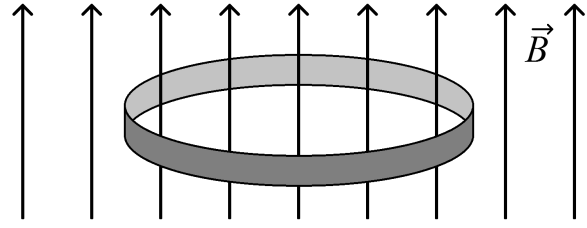
Using the Right-Hand Rule with Ampere's Law, with the current coming out of the page (thumb), the curled fingers show that the field above the center is

To the left.

III. (16 points) A circular loop made from a flexible conducting wire is shrinking. Its radius r as a function of time t is

$$r = r_0 \left(\pi^2 - \frac{t}{T} \right)$$

where r_0 and T are positive constants. The loop is perpendicular to a steady, uniform magnetic field of magnitude B , as shown. What is the magnitude of the emf induced in the loop at time T ? Express your answer in terms of parameters defined in the problem, and physical or mathematical constants.



Use Faraday's Law,

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{A} = -\frac{d}{dt} \int B \cos \theta dA$$

The loop's area vector is parallel to the field, so $\cos \theta = 1$. The magnetic field is uniform over the loop and does not change with time, so

$$\mathcal{E} = -\frac{d}{dt} \left[B \int dA \right] = -\frac{d}{dt} [BA] = -B \frac{dA}{dt}$$

The area of the loop is $A = \pi r^2$, so

$$\mathcal{E} = -B \frac{d}{dt} [\pi r^2] = -B\pi \frac{d}{dt} \left[r_0 \left(\pi^2 - \frac{t}{T} \right) \right]^2 = -B\pi r_0^2 \left[2 \left(\pi^2 - \frac{t}{T} \right) \left(\frac{-1}{T} \right) \right] = \frac{2B\pi r_0^2}{T} \left(\pi^2 - \frac{t}{T} \right)$$

At time $t = T$,

$$\mathcal{E} = \frac{2B\pi r_0^2}{T} \left(\pi^2 - \frac{T}{T} \right) = \frac{2B\pi r_0^2}{T} (\pi^2 - 1)$$

2. (5 points) In the problem above, what is the direction, if any, of the current induced in the loop at time T ?

Use Lenz' Law. As the loop shrinks, there is less magnetic flux up the page through it. Current must flow to oppose this change, making more flux up the page. Using the Right-hand Rule, and upward field is produced by a counter-clockwise current.

The current is to the right on the front of the loop, counter-clockwise when viewed from above.

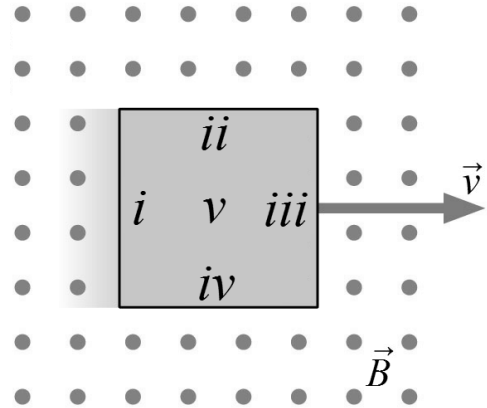
3. (5 points) A conducting plate moves to the right through a uniform magnetic field directed out of the page, as shown. Which location on the plate will be at greatest electric potential?

Consider a positive charge in the plate. As the plate moves, there will be a force

$$\vec{F} = q\vec{v} \times \vec{B}$$

down the page. The bottom edge of the plate will become positively charged and, since the plate must remain neutral overall, the top edge will become negatively charged. Electric field lines point from positive to negative charge, and also from high to low potential, so the greatest electric potential will be at

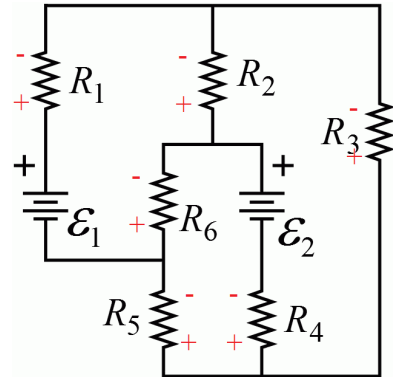
location *iv*, the bottom edge of the plate.



4. (5 points) Let the positive direction of current flow be up the page in the circuit shown. The current I_1 flows through resistor R_1 , etc. Which is a valid expression of Kirchoff's Loop Law?

As current flows from high to low potential through a resistor, positive current flow up the page makes the polarities of the resistors as marked. Only one of the offered choices is both a complete loop (that is, ends at its starting point) and has all the potential changes correct.

$$+\mathcal{E}_1 - I_1R_1 + I_3R_3 - I_4R_4 + \mathcal{E}_2 + I_6R_6 = 0$$



5. (5 points) The circuit shown consists of a battery with emf \mathcal{E} , an inductor with inductance L , two resistors each with resistance R , and a make-before-break switch. The switch has been in position “a” for a long time. What is the emf of the inductor immediately upon moving the switch to position “b”?

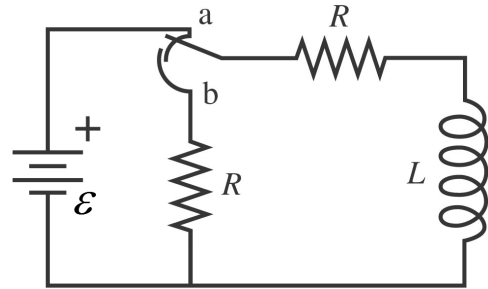
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After a long time, there are no changes in current flow, so there is no potential difference across the inductor. Applying Kirchhoff’s Loop Law to the outer loop,

$$+\mathcal{E} - IR = 0 \quad \Rightarrow \quad I = \mathcal{E}/R$$

This current also flows through the inductor. When the switch is thrown to “b”, the inductor will keep that same current flowing for the first instant. However, the current now must flow through two resistors. Applying Kirchhoff’s Loop Law to the loop on the left,

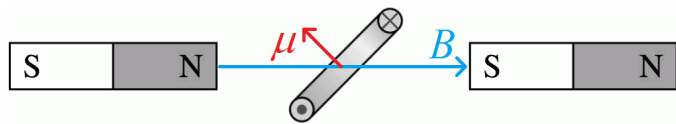
$$\Delta V_L - IR - IR = 0 \quad \Rightarrow \quad \Delta V_L = 2IR = 2(\mathcal{E}/R) = 2\mathcal{E}$$



6. (5 points) The loop, shown in cross-section, has a current into the page at the top, and out of the page at the bottom. It is held centered between two permanent magnets, as shown. How, if at all, does the loop move upon release?

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The field due to the permanent magnets is directed to the right, from North to South, as indicated. By the Right-Hand-Rule, the magnetic moment of the loop point up and to the left, as shown. The torque on the loop



$$\vec{\tau} = \vec{\mu} \times \vec{B}$$

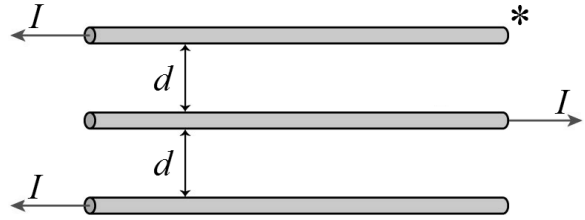
is into the page, and tends to cause the magnetic moment to align with the field, so

It rotates clockwise.

7. (5 points) Each of the three long straight wires carries the same current magnitude. What is the direction, if any, of the net force on the topmost wire, marked with an asterisk?

Parallel current attract, and anti-parallel currents repel. As the middle wire, producing a repelling force, is closer to the top wire than the bottom wire (producing an attracting force) is, the repelling force is of greater magnitude, and the net force is

Up the page.



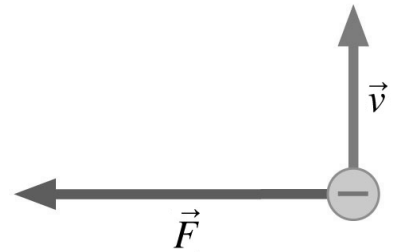
8. (5 points) An electron moves with velocity \vec{v} perpendicular to a magnetic field and experiences a force \vec{F} , as shown. What is the direction of the field?

The magnetic force on a moving charge is

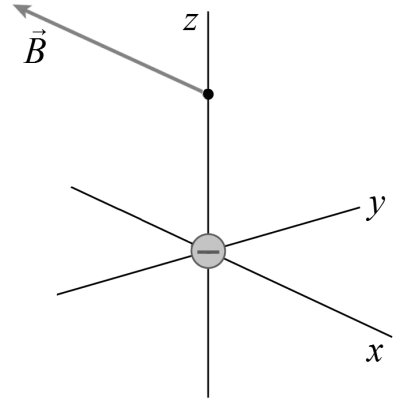
$$\vec{F} = q\vec{v} \times \vec{B}$$

As electrons are negative, this means $\vec{v} \times \vec{B}$ must be opposite the direction of \vec{F} , or to the right. With \vec{v} up the page, the Right-Hand Rule shows us that the field must be

Out of the page.



9. (5 points) As an electron passes through the origin, it generates a magnetic field on the $+z$ axis that points in the $-x$ direction, as shown. In what direction could this negatively-charged particle be traveling?



Use the Biot-Savart Law.

$$\vec{B} = \frac{\mu_0 q}{4\pi} \frac{\vec{v} \times \hat{r}}{|\vec{r}|^2}$$

Since the particle has a negative charge, the magnetic field must be opposite the direction of $\vec{v} \times \hat{r}$. The direction opposite the magnetic field is $+\hat{x}$. The direction of \hat{r} is $+\hat{z}$. As $\hat{j} \times \hat{z} = \hat{x}$, the particle could be traveling

In the $+y$ direction.

10. (5 points) What is the charge on the capacitor in the circuit shown, after the switch S has been closed for a very long time?

After a long time, the capacitor will be fully charged, so no current can be flowing through the resistor R on the right. Applying Kirchhoff's Loop Rule to the left loop,

$$+\mathcal{E} - I(3R) - I(2R) = 0 \quad \Rightarrow \quad I = \frac{\mathcal{E}}{5R}$$

so the potential across the center resistor $2R$ is

$$\Delta V_{2R} = I(2R) = \left(\frac{\mathcal{E}}{5R}\right)(2R) = \frac{2}{5}\mathcal{E}$$

That must also be the potential across the capacitor, as there is no potential difference across the resistor R on the right (since the current through it is zero). From the definition of capacitance,

$$Q = C \Delta V_C = C \Delta V_{2R} = 2C\mathcal{E}/5$$

