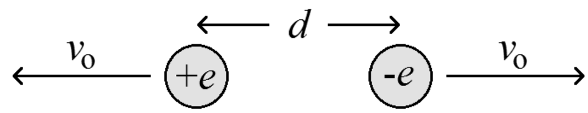
I. (18 points) A positron has the same mass as an electron, but has opposite charge. Consider a positron and an electron at rest, separated by a distance $d = 1.0\,\text{nm}$. What minimum velocity magnitude v_0 could be given to each particle, in opposite directions, so they move apart from each other and never return? If there is no such velocity magnitude because the particles will return for **any** v_0, prove it.

. . . .



Use the Work-Energy Theorem

$$W_{\text{ext}} = \Delta K + \Delta U + \Delta E_{\text{th}}$$

Choose a system consisting of the two particles. Once the initial velocity has been imparted, no external forces do work on the system. No non-conservative internal forces change the thermal energy of the system. The kinetic energy of the system is the sum of the kinetic energies of the two particles. The potential energy change of the system is due to the work done by the internal conservative electric force.

$$0 = \left(\tfrac{1}{2}m_1 v_{1f}^2 - \tfrac{1}{2}m_1 v_{1i}^2\right) + \left(\tfrac{1}{2}m_2 v_{2f}^2 - \tfrac{1}{2}m_2 v_{2i}^2\right) + \left(K\frac{q_1 q_2}{r_f} - K\frac{q_1 q_2}{r_i}\right) + 0$$

Note that the masses of the two particles are the same ($m_1 = m_2 = m_e$). If the particles are never to return, they must reach infinite separation ($r_f \to \infty$) with speeds of at least zero.

$$0 = \left(0 - \tfrac{1}{2}m_e v_0^2\right) + \left(0 - \tfrac{1}{2}m_e v_0^2\right) + \left(0 - K\frac{(+e)\,(-e)}{d}\right)$$

Solve for v_0.

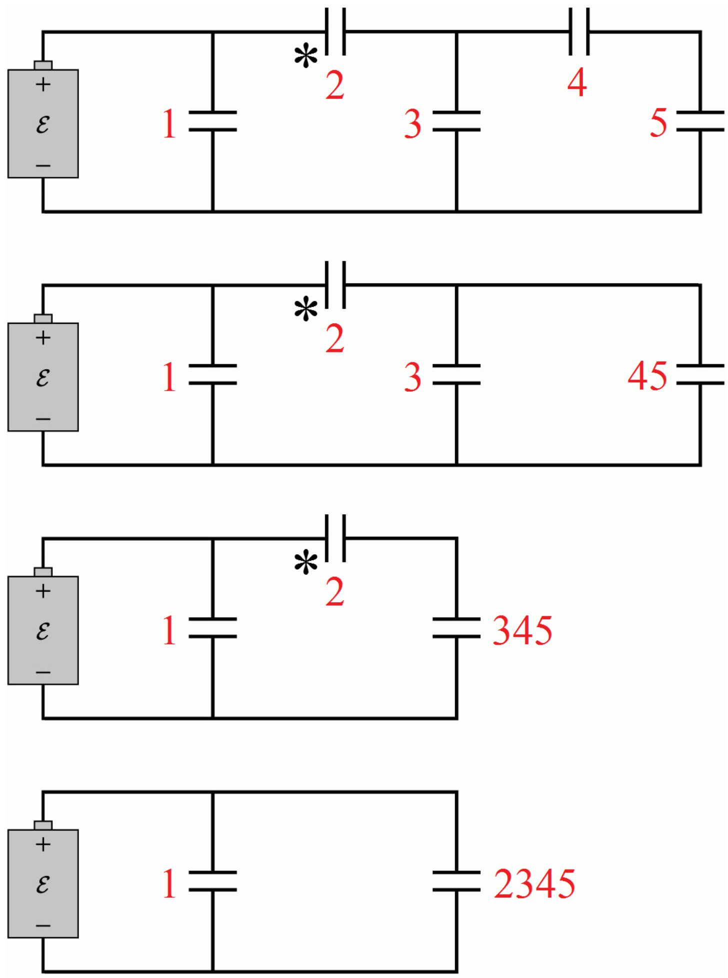
$$2\left(\tfrac{1}{2}m_e v_0^2\right) = K\frac{e^2}{d} \qquad \Rightarrow \qquad v_0 = e\sqrt{\frac{K}{m_e d}}$$

$$= \left(1.602\times10^{-19}\,\text{C}\right)\sqrt{\frac{8.988\times10^9\,\text{N}\cdot\text{m}^2/\text{C}^2}{\left(9.109\times10^{-31}\,\text{kg}\right)\left(1.0\times10^{-9}\,\text{m}\right)}}$$

$$= 5.0\times10^5\,\text{m/s}$$

II. (16 points) All capacitors in this circuit have identical capacitances C. The battery has a potential difference $\Delta V = \mathcal{E}$ between its terminals. What is the energy stored (with respect to zero in the uncharged state) in the capacitor marked with an asterisk? Express your answer in terms of parameters defined in the problem, and physical or mathematical constants.

. . . .



Find equivalent capacitances for combinations of capacitors until either the potential across or charge on the indicated capacitor has been determined.

The capacitors have been numbered for convenience. Capacitors 4 and 5 are in series.

$$C_{45} = \left(\frac{1}{C_4} + \frac{1}{C_5}\right)^{-1}$$

$$= \left(\frac{1}{C} + \frac{1}{C}\right)^{-1} = \left(\frac{2}{C}\right)^{-1} = C/2$$

Capacitors 3 and 45 are in parallel.

$$C_{345} = C_3 + C_{45} = C + C/2 = 3C/2$$

Capacitors 2 and 345 are in series.

$$C_{2345} = \left(\frac{1}{C_2} + \frac{1}{C_{345}}\right)^{-1} = \left(\frac{1}{C} + \frac{2}{3C}\right)^{-1}$$

$$= \left(\frac{5}{3C}\right)^{-1} = 3C/5$$

The potential across capacitor 2345 is the emf of the battery, $\mathcal{E}$, so from the definition of capacitance the charge on capacitor 2345 is

$$Q_{2345} = C_{2345}\,\Delta V_{2345} = \left(\frac{3C}{5}\right)\mathcal{E} = 3C\mathcal{E}/5$$

As capacitor 2 is in series with capacitor 345, it must carry this same charge.

$$Q_2 = Q345 = Q_{2345} = 3C\mathcal{E}/5$$

So the energy stored in capacitor 2 (the asterisked capacitor) is

$$U_2 = \frac{Q_2^2}{2C_2} = \frac{(3C\mathcal{E}/5)^2}{2C} = \frac{9}{50}C\mathcal{E}^2$$

1. (5 points) In the problem above, let the energy you found stored in the asterisked capacitor be U_0. If the battery were replaced with one that had a potential difference $2\mathcal{E}$ between its terminal, what energy U would now be stored that capacitor?

. . . .

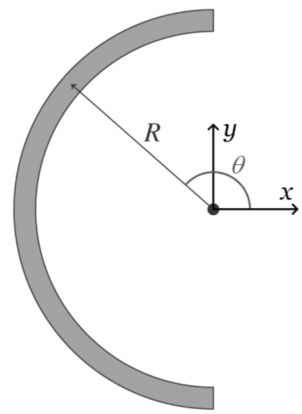
The energy in the asterisked capacitor must be proportional to $\mathcal{E}^2$ for the dimensions to be correct.

$$U = 4U_0$$

III. (16 points) A rod of length L and cross-sectional area A is bent into a semi-circle about the origin. The rod has non-uniform volume charge density

$$\rho = \rho_0 \theta$$

where ρ_0 is a positive constant, and θ is measured in radians from the $+x$ axis toward the $+y$ axis as shown. What is the electric potential at the origin, due to the rod, with respect to zero at infinity? Express your answer in terms of parameters defined in the problem, and physical or mathematical constants.



. . . .

Divide the rod into point-like bits, each with charge dq and volume dV_{ol}. This volume is the cross-sectional area A times a bit of arc length ds. then from the definition of the radian, $ds = R\,d\theta$. Add up the contribution due to all the bits of charge (that is, integrate).

$$V = \int dV = \int K\frac{dq}{r} = K\int \frac{\rho\, dV_{\text{ol}}}{R} = \frac{K}{R}\int \rho_0 \theta A\, ds = \frac{KA\rho_0}{R}\int_{\pi/2}^{3\pi/2} \theta R\, d\theta = KA\rho_0 \int_{\pi/2}^{3\pi/2} \theta\, d\theta$$

$$= KA\rho_0 \frac{\theta^2}{2}\bigg|_{\pi/2}^{3\pi/2} = \frac{KA\rho_0}{2}\left[\left(\frac{3\pi}{2}\right)^2 - \left(\frac{\pi}{2}\right)^2\right] = \frac{KA\rho_0}{2}\left[\left(\frac{9\pi^2}{4}\right) - \left(\frac{\pi^2}{4}\right)\right] = \frac{KA\rho_0}{2}\left[\frac{8\pi^2}{4}\right] = KA\rho_0\pi^2$$

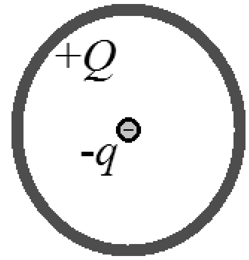
2. (5 points) What is the direction of the electric potential at the origin?

. . . .

Electric potential is a scalar!

The electric potential at the origin has no direction.

3. (5 points) A uniformly charged ring has radius R and total charge Q. With respect to zero at infinite separation, what is the electric potential energy U of a system consisting of this ring and a $-q$ point charge at its center?



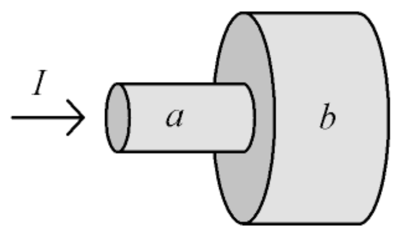
. . . .

The entire charge Q on the ring is the same distance from the center. As potential is a scalar, the potential at the center of the ring is the same as it would be if that entire charge Q were concentrated at a point. Therefore, the potential energy of this system consisting of a ring and point is the same as that of a system consisting of two points.

$$U = K\frac{q_1 q_2}{r} = -K\frac{Qq}{R}$$

4. (5 points) Current flows to the right through the wire shown. Segment b on the right has three times the diameter of segment a on the left. Segment b on the right has one half the conductivity of segment a on the left. Compare the electric field magnitude in segment b with that in segment a.

. . . .



The current must be the same in each segment, and can be related to the current density in each segment

$$I_a = I_b \qquad \Rightarrow \qquad J_a A_a = J_b A_b$$

Current density can be related to electric field.

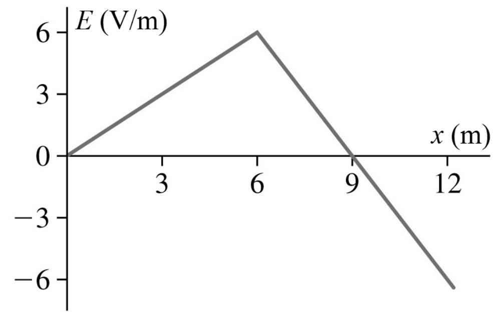
$$(\sigma_a E_a)\left(\frac{\pi}{4}d_a^2\right) = (\sigma_b E_b)\left(\frac{\pi}{4}d_b^2\right)$$

Solve for E_b.

$$E_b = \frac{\sigma_a}{\sigma_b}\left(\frac{d_a}{d_b}\right)^2 E_a = \frac{\sigma_a}{\frac{1}{2}\sigma_a}\left(\frac{d_a}{3d_a}\right)^2 E_a = \frac{2}{9}E_a$$

5. (5 points) A one-dimensional electric field points only in the $\pm x$ direction, with values shown in the graph. At which point in the range 0 m to 12 m does the electric potential have its maximum value?

. . . .



Electric field and potential difference are related by

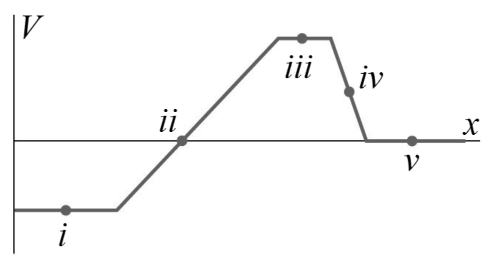
$$\Delta V = -\int \vec{E} \cdot d\vec{s}$$

so the difference in potential is represented by the opposite of the area under the curve on the graph. The area from 0 m to 9 m is positive. In this range, then, the potential has decreased from the value at 0 m. The potential increases between 9 m and 12 m, where the area under the curve is negative. However, this area has less magnitude than the area from 0 m to 9 m. By 12 m, the potential has still not increased enough to return to its value at 0 m. The maximum value, then, is found at

0 m

6. (5 points) A one-dimensional electric field points only in the $\pm x$ direction. The electric potential associated with the field is shown in the graph. At which of the indicated points does the electric field point in the negative direction with greatest magnitude (that is, have its most negative value)?

. . . .



Electric field and potential are related by

$$E_x = -\frac{\delta V}{\delta x}$$

so the electric field will have its most negative value at the point where the graph has its most positive slope,

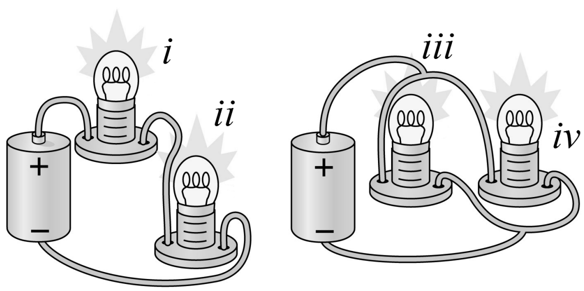
Point *ii*.

7. (5 points) Four identical light bulbs are connected to two identical batteries, as shown. Which bulb(s) is brightest?

. . . .

The entire potential difference created by the battery is across each of the bulbs *iii* and *iv*. However, that potential is across the two bulbs *i* and *ii* together, so only half the potential is across each of those bulbs. With more potential across a bulb, more current will flow through it, and it will be brighter.

Bulbs *iii* and *iv* are equally bright, and brighter than bulbs *i* and *ii*.



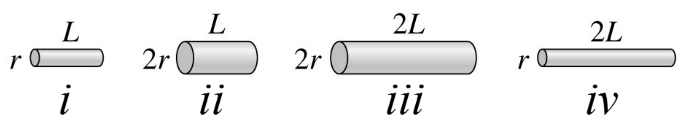
---

8. (5 points) Four wires are all made of the same material, but have different dimensions, as shown. Rank the resistances between the ends of the four wires from greatest to least.

. . . .

Resistance and resistivity are related by

$$R = \rho \frac{\ell}{A}$$



so

$$R_i = \rho \frac{L}{\pi r^2} \qquad R_{ii} = \rho \frac{L}{\pi (2r)^2} = \frac{1}{4} R_i \qquad R_{iii} = \rho \frac{2L}{\pi (2r)^2} = \frac{1}{2} R_i \qquad R_{iv} = \rho \frac{2L}{\pi (r)^2} = 2 R_i$$

meaning

$$iv > i > iii > ii$$

9. (5 points) A parallel plate capacitor is charged with a battery. This results in energy U_0 being stored in the capacitor (with respect to zero in the uncharged state). The battery **is disconnected** from the capacitor and insulating handles are used to push the plates closer together, until they have half their original spacing. How much energy U is stored in the capacitor, now?

. . . .

If the battery is disconnected from the capacitor, the charge on the plates cannot change. Remembering how the capacitance of a parallel plate capacitor depends on geometry,

$$U_0 = \frac{Q^2}{2C_0} = \frac{Q^2}{2\epsilon_0 A/d} = \frac{Q^2 d}{2\epsilon_0 A}$$

So

$$U = \frac{Q^2}{2C} = \frac{Q^2}{2\epsilon_0 A/\left(d/2\right)} = \frac{Q^2 d/2}{2\epsilon_0 A} = \frac{1}{2}\left[\frac{Q^2 d}{2\epsilon_0 A}\right] \qquad \Rightarrow \qquad U = U_0/2$$

10. (5 points) A parallel plate capacitor is charged with a battery. This results in energy U_0 being stored in the capacitor (with respect to zero in the uncharged state). The battery **remains connected** to the capacitor and insulating handles are used to push the plates closer together, until they have half their original spacing. How much energy U is stored in the capacitor, now?

. . . .

If the battery remains connected to the capacitor, the potential difference between the plates cannot change. Remembering how the capacitance of a parallel plate capacitor depends on geometry,

$$U_0 = \tfrac{1}{2} C_0 \left(\Delta V\right)^2 = \tfrac{1}{2}\epsilon_0 \frac{A}{d}\left(\Delta V\right)^2$$

So

$$U = \tfrac{1}{2} C \left(\Delta V\right)^2 = \tfrac{1}{2}\epsilon_0 \frac{A}{d/2}\left(\Delta V\right)^2 = 2\left[\tfrac{1}{2}\epsilon_0 \frac{A}{d}\left(\Delta V\right)^2\right] \qquad \Rightarrow \qquad U = 2U_0$$