PHYS 2211 Recitation 06 *Solution*

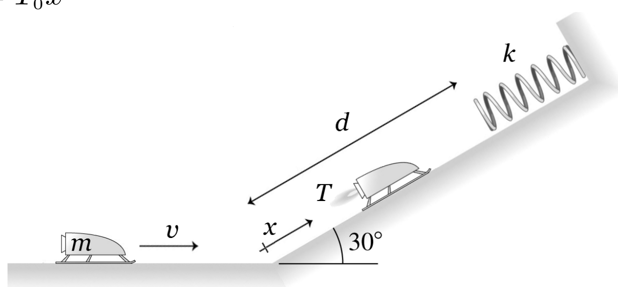
Jun 27–29

You should work in collaborative groups of 3–4, but each student must write up their own solution to the problem. Show all your work, and explain all your reasoning.

A rocket sled of mass $m = 98\,\text{kg}$ is coasting at speed $v = 9.8\,\text{m/s}$ along frictionless level ground with its engine off. When it encounters a frictionless hill that makes an angle of exactly $\theta = 30°$ above the horizontal, its engine comes on. As the engine warms up, it provides an increasing thrust magnitude T that varies with position x along the slope, according to

$$T = T_0 x$$

where $T_0 = 0.98\,\text{N/m}$. After sliding a distance $d = 9.8\,\text{m}$ along the slope, the sled encounters a spring with Hooke's Law Constant $k = 9.8\,\text{N/m}$. Assuming the mass of fuel consumed is negligible compared to the total mass of the sled, what is the maximum compression of the spring?



TA Analysis: Each student should complete this portion of the worksheet individually, following along as the TA works the problem. The work you show here will be factored into your grade!

A. Choose a physical principle with which to solve the problem. Why is this principle preferable to other principles?

↪ Use the energy principle. Newton's Second Law followed by constant-acceleration kinematics is inappropriate, as the varying spring force and thrust force will result in a non-constant acceleration. And using Newton's Second Law with a non-constant acceleration amounts to re-deriving the work and energy relationships.

B. Choose a system to analyze, and and note what forces are acting on it. Write an expression relating the system's energy change to the work done on it by those forces.

↪ We'll model the sled as a one-particle system. The Earth exerts a gravitational force and a normal force on the sled, and the spring will exert a force on it once contact is made. As the rocket sled pushes hot gasses out its exhaust, the hot gasses must exert a force on the sled (Newton's Third Law). From the energy principle,

$$\Delta E_{\text{sys}} = W_{\text{ext}} \qquad \Rightarrow \qquad \Delta K = W_g + W_n + W_{sp} + W_t$$

where the only energy change within the system is a kinetic energy change (as it must be for a one-particle system), and W_g, W_n, W_{sp}, W_t are the work done on the system by gravity, the normal force, the spring, and thrust, respectively.

C. Substitute an expression for the kinetic energy change of the sled. Are any terms necessarily zero?

↪ As the kinetic energy is $K = \frac{1}{2}mv^2$,

$$\tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2 = W_g + W_n + W_s + W_t$$

The sled must be stopped at the instant the spring has maximum compression, so $v_f = 0$. Also, the normal force always acts perpendicular to the surface on which the sled slides.

$$W_n = \int \vec{n} \cdot d\vec{s} = \int n \cos\phi \, ds = \int n \cos\left(90^\circ\right) \, ds = 0$$

so the normal force can do no work on the sled.

$$\tfrac{1}{2}m\left(0\right)^2 - \tfrac{1}{2}mv_i^2 = W_g + 0 + W_s + W_t \qquad \Rightarrow \qquad -\tfrac{1}{2}mv_i^2 = W_g + W_s + W_t$$

Student Analysis: Complete the worksheet in collaborative groups of 3–4, with each student writing up their own solution to the problem. Show all your work, and explain all your reasoning.

D. Let D be the maximum distance the spring compresses, which is the answer to the question. Find an algebraic expression for W_{sp}, the work done on the system by the spring. What does the sign of this work tell you?

↪ Hooke's Law tells us that the force exerted by the spring has magnitude $k\,\Delta s$, where Δs is the displacement of the spring's end. If we choose $s = 0$ at the free end of the spring, then $\Delta s = s$ and

$$W_{sp} = \int \vec{F}_{sp} \cdot d\vec{s} = \int F_{sp} \cos\phi \, ds = \int_0^D ks \cos\left(180^\circ\right) ds = -k\frac{s^2}{2}\Bigg|_0^D$$

$$= -\tfrac{1}{2}k\left[D^2 - \left(0\right)^2\right] = -\tfrac{1}{2}kD^2$$

where you'll note that the force of the spring is always *opposite* the displacement of the sled. The sign of W_{sp} is negative, as expected, as the force of the spring on the sled will reduce the sled's kinetic energy, thus bringing it to a stop.

E. Find an algebraic expression for W_t, the work done on the system by the rocket's thrust.

$\hookrightarrow$ From the bottom of the slope, to the point of maximum compression, the thrust will act for a distance $d + D$. The angle between the thrust and the displacement is always zero in this case.

$$W_t = \int \vec{T} \cdot d\vec{s} = \int T \cos\phi \, ds = \int_0^{d+D} T \cos 0 \, dx = \int_0^{d+D} T_0 x \, dx = T_0 \frac{x^2}{2}\bigg|_0^{d+D}$$

$$= \tfrac{1}{2} T_0 \left[(d+D)^2 - (0)^2 \right] = \tfrac{1}{2} T_0 \,(d+D)^2$$

F. Find an algebraic expression for W_g, the work done on the system by gravity. Substitute it, along with your expressions for the work done by the thrust and by the spring, into the relationship between work and kinetic energy found in part C.

$\hookrightarrow$ The sled must travel a distance $d + D$ along the slope.

$$W_g = \int m\vec{g} \cdot d\vec{s} = \int mg \cos\phi \, ds = \int_0^{d+D} mg \cos\left(90^\circ + \theta\right) \, dx = -mg \sin\theta \, x \bigg|_0^{d+D}$$

$$= -mg \sin\theta \,[(d+D) - 0] = -mg \sin\theta \,(d+D)$$

where we've made use of the fact that $\cos\left(90^\circ + \theta\right) = -\sin\theta$.

Substituting,

$$-\tfrac{1}{2} m v_i^2 = W_g + W_s + W_t = -mg \sin\theta \,(d+D) - \tfrac{1}{2} k D^2 + \tfrac{1}{2} T_0 \,(d+D)^2$$

Checkpoint: Before proceeding further, have your TA review your group's work so far.

G. The expression you found in part F is quadratic in D, the maximum compression of the spring. Rearrange it to the form

$$AD^2 + BD + C = 0$$

↪ Start with multiplying by 2.

$$-mv_i^2 = -2mg\sin\theta\,(d+D) - kD^2 + T_0\,(d+D)^2$$

$$T_0\,(d+D)^2 - kD^2 - 2mg\sin\theta\,(d+D) + mv_i^2 = 0$$

$$T_0 d^2 + T_0 2dD + T_0 D^2 - kD^2 - 2mg\sin\theta\,d - 2mg\sin\theta\,D + mv_i^2 = 0$$

$$(T_0 - k)\,D^2 + (2T_0 d - 2mg\sin\theta)\,D + \left(T_0 d^2 + mv_i^2 - 2mg\sin\theta\,d\right) = 0$$

H. Find values for the coefficients A, B, and C from part G. Use them to find two values for D.

↪ From part G we see that

$$A = T_0 - k = 0.98\,\mathrm{N/m} - 9.8\,\mathrm{N/m} = -8.82\,\mathrm{N/m}$$

$$B = 2T_0 d - 2mg\sin\theta = 2\,(0.98\,\mathrm{N/m})\,(9.8\,\mathrm{m}) - 2\,(98\,\mathrm{kg})\left(9.8\,\mathrm{m/s^2}\right)\sin(30°) = -941\,\mathrm{N}$$

$$\begin{aligned} C &= T_0 d^2 + mv_i^2 - 2mgd\sin\theta \\ &= (0.98\,\mathrm{N/m})\,(9.8\,\mathrm{m})^2 + (98\,\mathrm{kg})\,(9.8\,\mathrm{m/s})^2 - 2\,(98\,\mathrm{kg})\left(9.8\,\mathrm{m/s^2}\right)(9.8\,\mathrm{m})\sin(30°) \\ &= 94.1\,\mathrm{N{\cdot}m} \end{aligned}$$

so

$$\begin{aligned} D &= \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{941\,\mathrm{N} \pm \sqrt{(-941\,\mathrm{N})^2 - 4\,(-8.82\,\mathrm{N/m})\,(94.1\,\mathrm{N{\cdot}m})}}{2\,(-8.82\,\mathrm{N/m})} \\ &= -107\,\mathrm{m} \quad \text{or} \quad 0.0999\,\mathrm{m} \end{aligned}$$

I. Which value found in part H is the actual maximum compression of the spring? Justify your answer.

↪ When we set the distance for which the thrust acts to $d+D$ in part E, and the distance traveled by the sled along the slope to $d+D$ in part F, we made D a positive number. Two two significant figures, the maximum compression of the spring is

$$D = 0.10\,\mathrm{m}$$