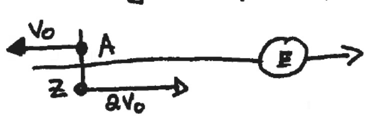
You should work in collaborative groups of 3–4, but each student must write up their own solution to the problem. Show all your work, and explain all your reasoning.

Captain Atomic is a distance $D = 3.0$ million km from Earth in his rocketship, drifting away from the planet with an initial speed $v_0 = 75\,\text{km/s}$. Suddenly, he is passed by the Death-Saucer of the nefarious Dr. Zed, coasting in the opposite direction at a fixed speed $2v_0$. Dr. Zed is going to use his Sonic Space-Ray™ to wipe out all life on Earth! Acting quickly, Captain Atomic engages his Oscillation Over-Thruster, to provide the acceleration needed to reverse course and overtake Dr. Zed before it is too late.

TA Analysis: Each student should complete this portion of the worksheet individually, following along as the TA works the problem. The work you show here will be factored into your grade!

A. **Organize:** Establish a coordinate system: choose an origin and identify the positive direction from that origin. Write out **symbolic**, *vector* expressions for (*i*) the position of the Earth and (*ii*) the initial positions and velocities of the two spacecraft. Identify other parameters (as yet unknown) that may also be needed to analyze the motion of the spacecraft. Clearly state any assumptions that you will make regarding the category of motion experienced by each spacecraft.

Choose origin at spaceships' initial location, let "to Earth" be the positive direction.



Earth: $\vec{X}_E = \langle +D \rangle$, $\vec{V}_E = \langle 0 \rangle$

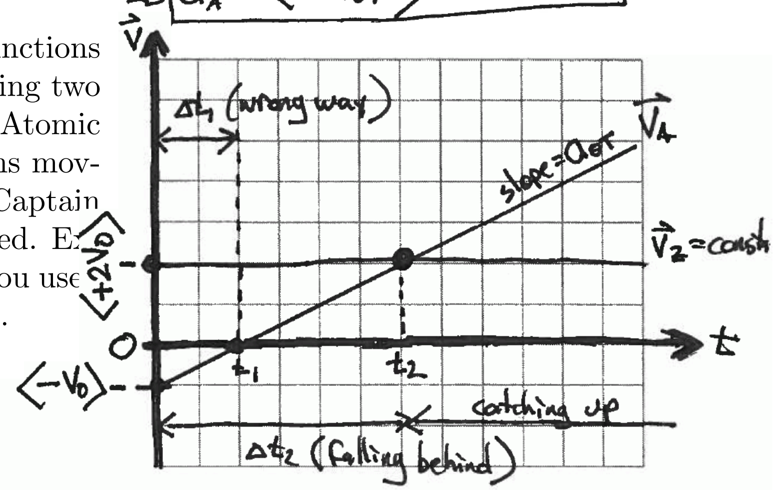
Atomic: $\vec{X}_{A0} = \langle 0 \rangle$ $\vec{V}_{A0} = \langle -V_0 \rangle$

Zed: $\vec{X}_{Z0} = \langle 0 \rangle$ $\vec{V}_{Z0} = \langle +2V_0 \rangle$

- assume no "time lag": Atomic instantly engages overthruster, when they pass each other
- Zed: uniform motion, $\boxed{\vec{V}_Z = \text{constant}}$
- Atomic uniform acceleration, magnitude unknown
  New parameter: a_{OT}

$\Rightarrow \boxed{\vec{a}_A = \langle +a_{OT} \rangle = \text{constant}}$

B. Graph the velocities of both craft as functions of time. Indicate on the graph the following two times: (*i*) the moment t_1 that Captain Atomic stops moving away from Earth and begins moving toward it; (*ii*) the moment t_2 that Captain Atomic actually begins to overtake Dr. Zed. Explain below in words the reasoning that you use to identify these two points on the graph.



(i) As long as $\vec{V}_A$ is negative, A is moving the wrong way

$\boxed{\vec{V}_{A1} = 0 \text{ at turnaround time } t_1}$

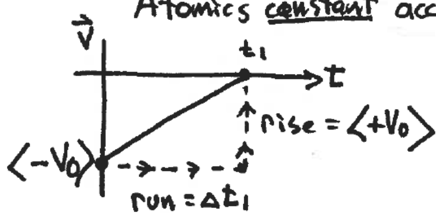
Even after turning around, A is still "losing ground", because Z is moving toward earth faster than A

(ii) A only starts to gain ground after his velocity exceeds Z's: "gaining" means $\vec{V}_A > \langle +2V_0 \rangle$

so: t_2 = moment velocities match: $\vec{V}_A = \vec{V}_Z = \langle +2V_0 \rangle$

C. Use graphical techniques to find a relationship between the magnitude of the Over-Thruster acceleration a_{OT}, Captain Atomic's initial speed v_0, and the elapsed time required for Captain Atomic to reach turnaround, $\Delta t_1 = t_1 - t_0$.

Atomic's constant acceleration is realized as the slope of his velocity graph:



$$\vec{a} = \frac{\Delta \vec{v}}{\Delta t} = \frac{\vec{V}_1 - \vec{V}_0}{t_1 - t_0} = \frac{0 - \langle -V_0\rangle}{\Delta t_{10}} = \langle + a_{OT}\rangle$$

so, $\boxed{V_0 = a_{OT}\,\Delta t_1}$

D. Use the graph to compare the elapsed times Δt_1 (to turnaround) and $\Delta t_2 = t_2 - t_0$ (to start catching up). Does your answer depend on the acceleration of Captain Atomic's rocketship? If so, how would doubling the acceleration affect the relative time intervals involved? If not, explain why the acceleration does not matter.

A simple graphical comparison of gridlines indicates $\boxed{\Delta t_2 = 3\Delta t_1}$

this conclusion is independent of the specific value of a_{OT}

why: to turnaround means: $\vec{V}_{A0} = \langle -V_0\rangle \to \vec{V}_{A1} = \langle 0\rangle \to \Delta\vec{V}_{01} = \langle +V_0\rangle$

starting to catch up: $\vec{V}_{A0} = \langle -V_0\rangle \to \vec{V}_{A2} = \langle +2V_0\rangle \to \Delta\vec{V}_{02} = \langle +3V_0\rangle$

since $t_0 \to t_2$ requires 3x the velocity change, it must require 3x the elapsed time

$\Delta\vec{v} = \vec{a}\Delta t$ $\quad \dfrac{\Delta\vec{V}_{02} = \vec{a}_{OT}\,\Delta t_2}{\Delta\vec{V}_{01} = \vec{a}_{OT}\,\Delta t_1} \to \dfrac{|\Delta\vec{V}_{02}|}{|\Delta\vec{V}_{01}|} = \dfrac{\Delta t_2}{\Delta t_1} = 3$ regardless of a_{OT}

E. What *mathematical constraint* is equivalent to the statement "Captain Atomic catches Dr. Zed"? Explain *in words* how you would evaluate this condition, working directly from the graph in Part B (that is, without invoking any "memorized kinematic equations"). How will you account for the initial interval Δt_1 , during which Captain Atomic is moving in the *wrong* direction?

"A catches Z" → at some time other than $t_0 = 0$, $\boxed{\vec{X}_A = \vec{X}_Z}$

since the both start at the same location, $\vec{X}_{A0} = \vec{X}_{Z0} = 0$, this condition can also be written in terms of displacements $\boxed{\Delta\vec{X}_A = \Delta\vec{X}_Z}$ defines "A catches Z"

(necessarily at some $\vec{x}_f \neq 0$)

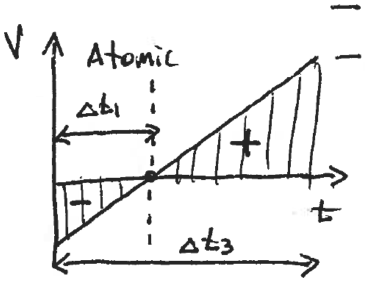
Stating constraint in terms of displacement allows us to invoke graphical rules: $\boxed{\text{displacement is found as area under velocity curve}}$

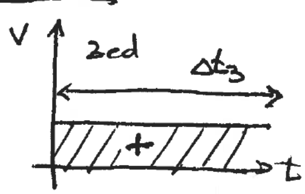
ie – require same net areas, for both A and Z, after some elapsed time $\Delta t_3 \neq 0$

"Wrong way motion" – negatively valued velocity

– area is technically "over" curve, not "under"

– $\boxed{\text{counts as a negative area}}$

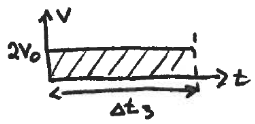




Student Analysis: Complete the worksheet in collaborative groups of 3–4, with each student writing up their own solution to the problem. Show all your work, and explain all your reasoning.

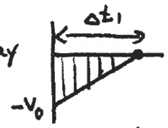
F. Let $\Delta t_3 = t_3 - t_0$ represent the total elapsed time for Captain Atomic to catch Dr. Zed. Follow the strategy laid out on Part E to write out a *detailed* equation that satisfies the necessary condition. Your expression should involve the parameters Δt_1, Δt_3, v_0, and $a_{\rm OT}$. (Some simple sketches—triangles and rectangles—might help!)

Zed first → he's easiest:



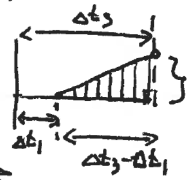
$\vec{\Delta X}_Z = \underbrace{\langle +2V_0 \rangle}_{\text{height}} \times \underbrace{\Delta t_3}_{\text{width}}$

Atomic → ~~wrong~~ wrong way



triangle $\vec{\Delta X}_{A-} = \frac{1}{2}\text{base}\cdot\text{height} = \frac{1}{2}(\Delta t_1)(-V_0) = -\frac{1}{2}V_0\Delta t_1$

→ right way



$\} \ \vec{\Delta V}_A = \vec{a}_{OT}(\Delta t_3 - \Delta t_1)$

$\vec{\Delta X}_{A+} = \frac{1}{2}(\Delta t_3 - \Delta t_1)\left[a_{OT}(\Delta t_3 - \Delta t_1)\right]$

$= +\frac{1}{2}a_{OT}(\Delta t_3 - \Delta t_1)^2$

so: require $\vec{\Delta X}_Z = \vec{\Delta X}_A$

$2V_0\Delta t_3 = -\frac{1}{2}a_{OT}\Delta t_1^2 + \frac{1}{2}a_{OT}(\Delta t_3 - \Delta t_1)^2 = \cancel{-\frac{1}{2}a_{OT}\Delta t_1^2} + \frac{1}{2}a_{OT}(\Delta t_3^2 + \cancel{\Delta t_1^2} - 2\Delta t_3\Delta t_1)$

$$\boxed{2V_0\Delta t_3 = \frac{1}{2}a_{OT}\Delta t_3^2 - a_{OT}\Delta t_3\Delta t_1}$$

Note: one can simply further by cancelling a factor of Δt_3

$$\boxed{2V_0 = \frac{1}{2}a_{OT}\Delta t_3 - a_{OT}\Delta t_1}$$

G. Use the results of Part C to eliminate the unknown v_0 from your expression in Part F. Look closely at your result—can you find any other factors that cancel out of all terms? Simplify your result, to find a direct relationship between Δt_3 and Δt_1.

Part C : $\boxed{V_0 = a_{OT}\Delta t_1}$ → $2\cancel{a_{OT}}\Delta t_1 = \frac{1}{2}\cancel{a_{OT}}\Delta t_3 - \cancel{a_{OT}}\Delta t_1$

⇒ Cool-a-mooly: a_{OT} drops out, too!

$2\Delta t_1 = \frac{1}{2}\Delta t_3 - \Delta t_1$

$3\Delta t_1 = \frac{1}{2}\Delta t_3$

$$\boxed{\Delta t_3 = 6\Delta t_1}$$

Checkpoint: Before continuing further, have the TA review your group's work so far.

H. Use the result of Parts F and G to write out an expression for Captain Atomic's *position* (not displacement) when he overtakes Dr. Zed. Express your answer **only** in terms of v_0 and a_{OT}. (Don't forget our conclusions in Part C!) By requiring this position to be *short* of Earth's position, determine a symbolic expression for the minimum possible acceleration magnitude for the Oscillation Over-Thruster, in terms of v_0 and D **only**.

since $\vec{\Delta X_A} = \vec{\Delta X_Z}$ at the moment of overtake, we could just use $\vec{\Delta X_Z}$, but instructions said "captain Atomic", so:

$$\vec{\Delta X_A} = \frac{1}{2} a_{OT} \Delta t_3^2 - a_{OT} \Delta t_3 \Delta t_1 \text{, evaluated } \underline{\text{when}}\ \Delta t_3 = 6\Delta t_1$$

$$\vec{\Delta X_{A,f}} = \frac{1}{2} a_{OT} (6\Delta t_1)^2 - a_{OT}(6\Delta t_1)\Delta t_1 = 18 a_{OT} \Delta t_1^2 - 6 a_{OT} \Delta t_1^2 = 12 a_{OT} \Delta t_1^2$$

$\underline{\text{but}}$: $V_0 = a_{OT} \Delta t_1 \rightarrow \Delta t_1 = V_0 / a_{OT}$ so: $\boxed{\vec{\Delta X_{A,f}} = 12 a_{OT} \left(\frac{V_0}{a_{OT}}\right)^2 = \left\langle +\frac{12 V_0^2}{a_{OT}} \right\rangle}$

"Not at Earth yet"

$$\vec{\Delta X_{A,f}} \lesssim \langle +D \rangle$$

$$\frac{12 V_0^2}{a_{OT}} \leq D \rightarrow a_{OT} \gtrsim \frac{12 V_0^2}{D} \rightarrow$$

minimum possible acceleration : use equality

$$\boxed{a_{min} = \frac{12 V_0^2}{D}}$$

I. **Assess:** *Before* you make any numerical substitutions, confirm that your symbolic expression in Part H has the correct physical dimension for describing *acceleration*. After doing so, substitute the numerical data at the beginning of the problem to determine a *minimum* possible value for the acceleration a_{OT}. While you're at it, find the (maximum) values for the three elapsed times: Δt_1 (elapsed time to turnaround), Δt_2 (elapsed time until Captain Atomic *starts* catching up to Dr. Zed), and Δt_3 (elapsed time until Captain Atomic actually *reaches* Dr. Zed), at minimum acceleration.

dimensional analysis : $\frac{12 V_0^2}{D} \rightarrow \frac{[\text{length}/\text{time}]^2}{[\text{length}]} = \frac{[\text{length}]^{\cancel{2}}}{\cancel{[\text{length}]}[\text{time}]^2} = \boxed{\frac{[\text{length}]}{[\text{Time}]^2} \text{ correct units for acceleration}}$

Plugging in

$$a_{OT} = \frac{12[75 \text{km/s}]^2}{[3.0 \times 10^6 \text{km}]} = 0.0225 \frac{\text{km}}{\text{s}^2} \times \frac{1000 \text{m}}{1 \text{Km}}$$

$$\boxed{a_{OT,min} = 22.5 \text{m/s}^2} \ (\approx 2.3 \text{ gee's})$$

$$\Delta t_1 = \frac{V_0}{a_{OT}} = \frac{V_0}{\frac{12 V_0^2}{D}} = \frac{D}{12 V_0} \rightarrow \Delta t_1 = \frac{3e6 \text{ km}}{12(75 \text{ km/s})} = \boxed{3300 \text{ sec}} = 0.93 \text{ hours}$$

two-digit precision

$$\Delta t_2 = 3\Delta t_1 = 3 \frac{D}{12 V_0} = \frac{D}{4 V_0}$$

$\Delta t_2 = 10{,}000 \text{ sec} = 2.8 \text{ hours}$

$\boxed{\Delta t_2 = 1.0 \times 10^4 \text{sec}}$ two-digit precision

$$\Delta t_3 = 6\Delta t_1 = \frac{D}{2} = \boxed{2.0 \times 10^4 \text{ sec}} = 5.6 \text{ hours}$$