
I. (16 points) A 0.50 kg object moves along the x axis. It is traveling at $+1.6\,\mathrm{m/s}$ when it reaches $x = 1.0\,\mathrm{m}$ and a force

$$\vec{F} = \frac{2.0\,\mathrm{N{\cdot}m^2}}{x^2}\,\hat{\imath}$$

begins to act on it. What is the velocity of the object when it reaches $x = 3.0\,\mathrm{m}$?

. . . .

The work done on a particle changes its kinetic energy. Let $F_0 = 2.0\,\mathrm{N{\cdot}m^2}$.

$$\Delta K = W_{\text{ext}} = \int \vec{F}\cdot d\vec{s} = \int_{x_i}^{x_f} F_x\,dx = \int_{x_i}^{x_f} \frac{F_0}{x^2}\,dx = \left.\frac{-F_0}{x}\right|_{x_i}^{x_f} = F_0\left(\frac{1}{x_i} - \frac{1}{x_f}\right)$$

$$= \tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2 = F_0\left(\frac{1}{x_i} - \frac{1}{x_f}\right) \qquad \Rightarrow \qquad v_f^2 = \frac{2F_0}{m}\left(\frac{1}{x_i} - \frac{1}{x_f}\right) + v_i^2$$

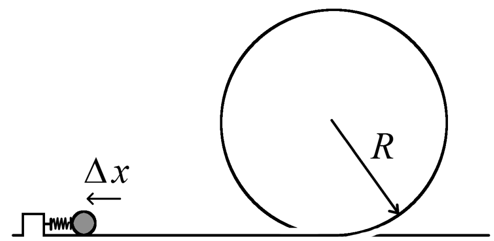
so

$$v_f = \sqrt{\frac{2F_0}{m}\left(\frac{1}{x_i} - \frac{1}{x_f}\right) + v_i^2} = \sqrt{\frac{2\,(2.0\,\mathrm{N{\cdot}m^2})}{0.50\,\mathrm{kg}}\left(\frac{1}{1.0\,\mathrm{m}} - \frac{1}{3.0\,\mathrm{m}}\right) + (1.6\,\mathrm{m})^2} = 2.8\,\mathrm{m/s}$$

As this is positive, it represents a velocity in the $+x$ direction.

$$+2.8\,\hat{\imath}\,\mathrm{m/s}$$

II. (16 points) A small ball with mass m is pushed against a spring with spring constant k. The ball is then released to travel on a frictionless loop-the-loop track with radius R. What is the minimum compression of the spring Δx for the ball stay in contact with the track all the way around? Express your answer in terms of parameters defined in the problem, and physical or mathematical constants. (*On Earth.*)



. . . .

First, find the speed required for the ball to remain in contact with the track at the very top. Use Newton's Second Law. Sketch a Free Body Diagram of the ball. There will be a normal force $\vec{n}$ and a gravitational force $m\vec{g}$ downward acting on it. Choose a coordinate system. I'll choose the $+c$ direction downward, toward the center of the loop, which is the direction of the known acceleration. When writing Newton's Second Law, I'll show signs explicitly, so symbols represent magnitudes.

$$\sum F_c = n + mg = ma_c = m\frac{v^2}{r} \qquad \Rightarrow \qquad v^2 = n\frac{m}{r} + gr$$

At the minimum speed to loop successfully, the normal force becomes zero for just an instant at the top of the loop. The track doesn't push down on the ball, but the ball doesn't lose contact with the track. The radius of the ball's motion is the radius of the loop.

$$n = 0 \qquad \Rightarrow \qquad v^2 = gR$$

Now use the Energy Principle to determine the compression of the spring that will result in this speed at the top of the loop. Choose a system consisting of the ball, the Earth, and the spring. With this choice, no external forces do work on the system (the normal force from the track is always perpendicular to the ball's velocity, and so does no work). There's a potential energy due to the internal conservative spring force, and another potential energy due to the internal conservative gravitational force. The kinetic energy of the ball changes, but the kinetic energy change of the Earth is negligible. The track is frictionless, so there is no thermal energy change.

$$W_{\text{ext}} = \Delta K + \Delta U + \Delta E_{\text{th}} \qquad \Rightarrow \qquad 0 = \Delta K + \Delta U_s + \Delta U_g + 0$$
$$= \left(\tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2\right) + \left(\tfrac{1}{2}ks_f^2 - \tfrac{1}{2}ks_i^2\right) + (mgh_f - mgh_i)$$

The initial speed of the ball is zero, and the final speed was found above. The final compression of the spring is zero, and the initial compression is the answer to the question. If we let the initial height of the ball be zero, the final height of the ball is $2R$.

$$0 = \left(\tfrac{1}{2}m\,[gR] - 0\right) + \left(0 - \tfrac{1}{2}k\,[\Delta x]^2\right) + (mg\,[2R] - 0)$$

Solve for the initial compression of the spring.

$$\tfrac{1}{2}k\,[\Delta x]^2 = \tfrac{1}{2}mgR + 2mgR \qquad \Rightarrow \qquad k\,[\Delta x]^2 = 5mgR \qquad \Rightarrow \qquad \Delta x = \sqrt{5mgR/k}$$

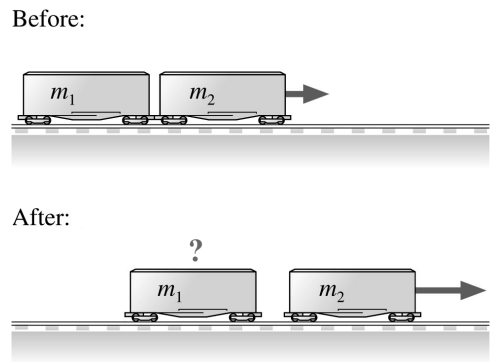
1. (6 points) In the problem above, the ball falls off the track before it reaches the top if the spring is compressed $\Delta x'$. How does $\Delta x'$ compare to Δx found above?

. . . .

If the ball falls off the track, it must be moving too slowly to remain in contact. Its kinetic energy is less than in the problem above, so the initial potential energy stored in the spring must be less than in the problem above.

$$0 < \Delta x' < \Delta x$$

III. (16 points) Coupled railcars with masses m_1 and m_2 are coasting along level frictionless rails at 12 m/s to the right, as shown. A chemical explosion separates the cars, after which the car with mass m_2 is found to be traveling at 18 m/s to the right. If $m_1 = 15{,}000\,\text{kg}$ and $m_2 = 25{,}000\,\text{kg}$, what is the resulting **velocity** of the car with mass m_1? (*On Earth.*)



A system consisting of the railcars had no net external force on it. Momentum is conserved in that system.

$$\vec{P}_f = \vec{P}_i$$

Letting the positive direction be to the right, directions can be shown with signs in one dimension.

$$m_1 v_{1f} + m2v_{2f} = (m_1 + m_2)\, v_i \qquad \Rightarrow \qquad v_{if} = \frac{(m_1 + m_2)\, v_i - m2v_{2f}}{m_1}$$

So

$$v_{1f} = \frac{(15{,}000\,\text{kg} + 25{,}000\,\text{kg})\,(12\,\text{m/s}) - (25{,}000\,\text{kg})\,(18\,\text{m/s})}{15{,}000\,\text{kg}} = 2.0\,\text{m/s}$$

This positive value means the velocity is

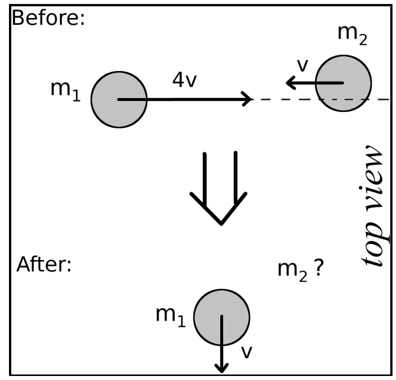
2.0 m/s to the right

2. (6 points) In the problem above, if, instead of a chemical explosion, the cars were separated by compressed air, or a big spring, how would the resulting velocity of the car with mass m_1 be affected? Assume the initial velocity of the cars, and the final speed of the car with mass m_2, are the same as stated above.

Internal forces, whatever their nature, cannot change the total momentum of a system. Since the final speed of the car with mass m_2 is the same,

All three separation methods result in the same speed for the m_1 car.

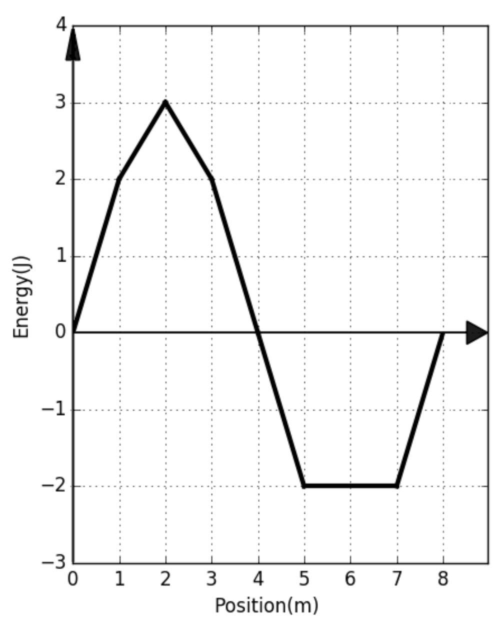
3. (8 points) Two identical balls have a collision on a frictionless table as shown in a top view. The initial speeds of m_1 and m_2 are $4v$ and v respectfully. After the collision, the velocity of m_1 is shown, what is the direction of the velocity of m_2?



. . . .

The collision cannot change the total momentum of the two-ball system. Before the collision, m_1 has momentum $4mv$ to the left, and m_2 has momentum mv to the right, so the total momentum of the system is $3mv$ to the right. After the collision, m_1 has momentum mv down the page. If the total is to remain $3mv$ to the right, $m2$ must have momentum $3mv$ to the right and mv up the page.

4. (8 points) The potential energy of a system depends on the position of an object within the system as shown. In which range will maximum force be on the object?



. . . .

As $F_x = -dU/dx$, the maximum force will be where the potential energy graph has the most negative slope. Among the offered choices, that is

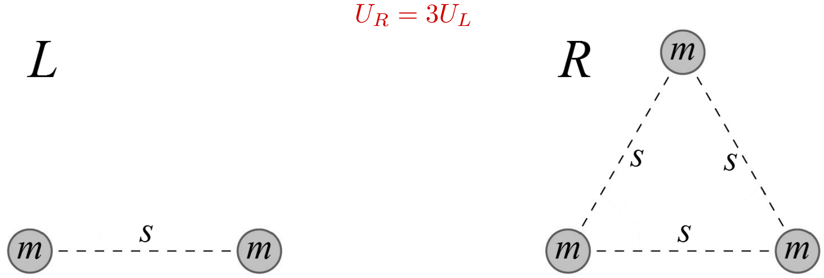
3 to 4 meters

5. (8 points) The system on the left (L) consists of two particles with mass m separated by a center-to-center distance s. The system on the right (R) consists of three particles with mass m arranged on the vertexes of an equilateral triangle with sides of length s. With respect to zero at infinite separation, compare the gravitational potential energy U_R of the system on the right, with U_L, the gravitational potential energy of the system on the left.

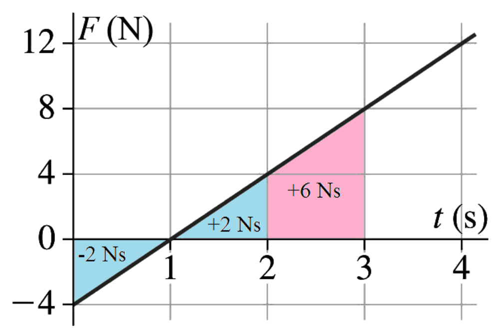
. . . .

For each pair of particles there is potential energy $U = -Gm_1m_2/r = -Gm^2/s$ in the system. The system on the left has one pair of particles, while the system on the right has three pairs of particles.

$U_R = 3U_L$



6. (8 points) A 2.0 kg object is travelling in the positive direction at 5.0 m/s when it becomes subject to the force depicted in the graph at time $t = 0$ s. What is the velocity of the object at time $t = 3.0$ s?



. . . .

The impulse on an object changes its momentum, and is the area under a force-time curve.

$$\vec{J} = \int \vec{F}\, dt = \Delta\vec{p} = m\vec{v}_f - m\vec{v}_i$$

The area under this force-time curve from zero to $t = 3.0$ s is $+6$ N·s. Letting signs indicate directions in one dimension,

$$v_f = \frac{J}{m} + v_i = \frac{+6\,\text{N·s}}{2.0\,\text{kg}} + 5.0\,\text{m/s} = 8.0\,\text{m/s}$$

7. (8 points) A block with kinetic energy K is sliding along a horizontal table top, then comes to a stop in a distance d due to friction. If the friction force on the block has magnitude f, ...

. . . .

The *kinetic* energy of the block changes by $-K$ as the block comes to a stop, but the *total* energy of the block doesn't change that much, as the thermal energy of the block increases. However, the thermal energy of the block doesn't increase by the entire K, as there is a thermal energy increase in the table top, as well. So ...

the kinetic energy of the block changes by $-K$ while
the thermal energy of the block and table top increases by fd.