
I. (16 points) An object starts from rest at the origin at time $t = 0\,\mathrm{s}$, then moves along the x axis with acceleration that depends on time according to

$$a_x(t) = 6.0\,\mathrm{m/s^4}\,t^2$$

What is the object's *average acceleration* between time $t = 1.0\,\mathrm{s}$ and $t = 3.0\,\mathrm{s}$?

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The definition of average acceleration is

$$\vec{a}_{\mathrm{avg}} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_f - \vec{v}_i}{t_f - t_i}$$

so an expression for velocity is needed. (As the object moves in only one dimension, vector notation will be dropped, and directions will be indicated by the signs.) Let the constant $6.0\,\mathrm{m/s^4}$ be represented by A.

$$a = \frac{dv}{dt} \qquad \text{so} \qquad v = \int a\,dt = \int A\,t^2\,dt = \frac{A}{3}\,t^3 + B$$

where B is an integration constant. As the object starts at rest at time $t = 0\,\mathrm{s}$,

$$0 = \frac{A}{3}\,(0\,\mathrm{s})^3 + B \qquad \Rightarrow \qquad B = 0\,\mathrm{m/s} \qquad \text{so} \qquad v = \frac{A}{3}\,t^3$$

Then

$$v(1\,\mathrm{s}) = \frac{6.0\,\mathrm{m/s^4}}{3}\,(1.0\,\mathrm{s})^3 = 2.0\,\mathrm{m/s} \qquad \text{and} \qquad v(3\,\mathrm{s}) = \frac{6.0\,\mathrm{m/s^4}}{3}\,(3.0\,\mathrm{s})^3 = 54\,\mathrm{m/s}$$

so

$$a_{\mathrm{avg}} = \frac{v_f - v_i}{t_f - t_i} = \frac{54\,\mathrm{m/s} - 2.0\,\mathrm{m/s}}{3.0\,\mathrm{s} - 1.0\,\mathrm{s}} = +26\,\mathrm{m/s}$$

II. (16 points) An angry, ball-shaped animal is trying to attack a target on a building 38 meters high, as shown. By using a slingshot, it can fire a ball with fixed initial speed $v_0 = 35\,\mathrm{m/s}$ and launch angle $\theta = 55°$, so the ball comes down to land on the target. What is the horizontal range R to the target? (*On Earth*.)

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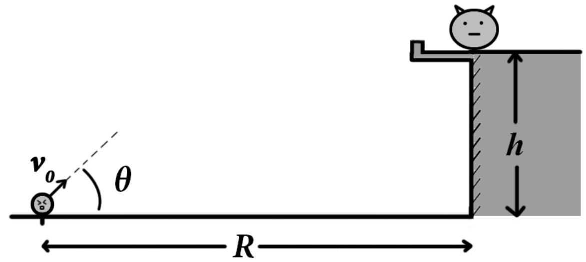

This is a projectile motion problem, that is, a two-dimensional constant-acceleration kinematics problem in which the horizontal acceleration is zero and the vertical acceleration is g downward.

Let the origin be at the launch point, with the $+y$ direction upward, and the $+x$ direction to the right. The time required for the ball to reach the target can be determined from the vertical information.

$$y = y_0 + v_{0y}\,\Delta t + \tfrac{1}{2}a_y\,(\Delta t)^2 \qquad \Rightarrow \qquad \tfrac{1}{2}a_y\,(\Delta t)^2 + v_{0y}\,\Delta t + (y_0 - y) = 0$$

is quadratic in Δt, so

$$\Delta t = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \qquad \text{where} \qquad \begin{aligned} A &= \tfrac{1}{2}a_y = \tfrac{1}{2}\left(-9.8\,\mathrm{m/s^2}\right) = -4.9\,\mathrm{m/s^2} \\ B &= v_{0y} = v_0 \sin\theta = (35\,\mathrm{m/s})\sin 55° = 28.6\,\mathrm{m/s} \\ C &= y_0 - y = y_0 - h = 0\,\mathrm{m} - 38\,\mathrm{m} = -38\,\mathrm{m} \end{aligned}$$

So

$$\Delta t = \frac{-\left(28.6\,\mathrm{m/s}\right) \pm \sqrt{\left(28.6\,\mathrm{m/s}\right)^2 - 4\left(-4.9\,\mathrm{m/s^2}\right)\left(-38\,\mathrm{m}\right)}}{2\left(-4.9\,\mathrm{m/s^2}\right)} = 2.03\,\mathrm{s} \quad \text{or} \quad 3.82\,\mathrm{s}$$

The ball, then, is at height h twice, once on the way up, and once on the way down. The problem specifies that the target is hit when the ball is coming down, so choose the longer time, 3.82 s.

Now consider the horizontal direction. Remember that the x component of the acceleration is zero.

$$x = x_0 + v_{0x}\,\Delta t + \tfrac{1}{2}a_x\,(\Delta t)^2 \qquad \Rightarrow \qquad R = v_0 \cos\theta\,\Delta t = (35\,\mathrm{m/s})\cos 55°\,(3.82\,\mathrm{s}) = 77\,\mathrm{m}$$

1. (6 points) In what direction is the ball travelling when it hits the target?

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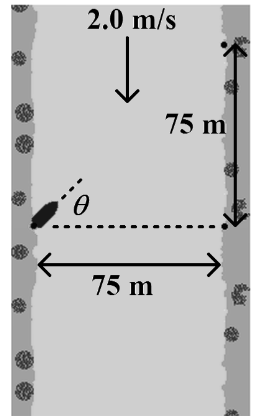
One could calculate the horizontal and vertical components of the velocity, then use the inverse tangent to calculate the angle at which the ball is travelling. However, this is a multiple-choice question. The problem specifies that the ball is coming down when it hits the target (that is, travelling in a direction below the horizontal). From the symmetry of a parabola, it would be travelling at 55° below the horizontal when it returns to ground level. When above that, it must be travelling at angles less than 55°. The only choice offered that satisfies these conditions is

24° below horizontal.

III. (16 points) A powerboater wants to cross a river that is flowing 2.0 m/s down the page. The river is 75 m wide, and the boater wants to land at a point 75 m upstream from the starting point. If the boat can move at 6.3 m/s through the water, at what angle θ upstream from straight across should the bow of the boat be aimed? *Hint:* You **shouldn't** need to know the identity $2 \sin\phi \cos\phi = \sin 2\phi$, but there are situations in which it may be useful.



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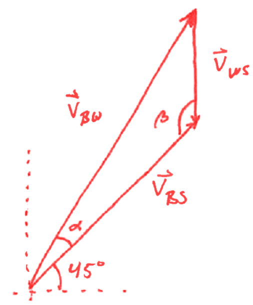
As the boat is carried downstream while it moves through the water, the velocity of the boat with respect to the shore ($\vec{v}_{\rm BS}$) is the sum of the velocity of the boat through the water ($\vec{v}_{\rm BW}$) and the velocity of the water with respect to the shore ($\vec{v}_{\rm WS}$),

$$\vec{v}_{\rm BS} = \vec{v}_{\rm BW} + \vec{v}_{\rm WS}$$

This vector sum can be represented graphically, as shown. The magnitudes $v_{\rm BW}$ and $v_{\rm WS}$ are known. The angle β can be determined to be 135°. The Law of Sines can be used to find the angle α.



$$\frac{\sin\alpha}{v_{\rm WS}} = \frac{\sin\beta}{v_{\rm BW}} \qquad \Rightarrow \qquad \alpha = \sin^{-1}\left[\sin\beta \, \frac{v_{\rm WS}}{v_{\rm BW}}\right] = \sin^{-1}\left[\sin 135° \, \frac{2.0\,\text{m/s}}{6.3\,\text{m/s}}\right] = 13°$$

This is the angle in addition to the 45° at which the boat is to travel with respect to the shore, so

$$\theta = 13° + 45° = 58°$$

2. (6 points) If, instead, the boater wished to land at a point 75 m *downstream* from straight across, the bow of the boat must be aimed at an angle ϕ downstream from straight across. How are the magnitudes of ϕ and θ related?

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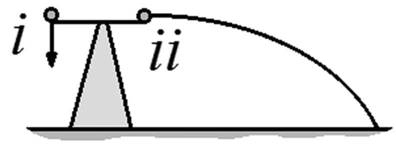
For the boat to travel upstream at 45°, the bow must be aimed at an angle θ greater than 45°. However, for the boat to travel downstream at 45°, the bow must be aimed at an angle ϕ less than 45°. Therefore,

$$\phi < \theta$$

3. (8 points) A device atop a tower drops an object (i) while simultaneously throwing an identical object (ii) horizontally. Which object, if either, reaches the level ground first? Which object, if either, strikes the ground with greater speed? (*On Earth.*)



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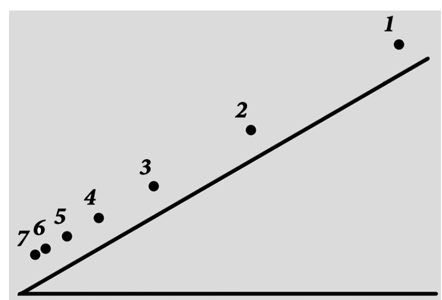
The vertical motions of the two objects are identical. Therefore, the vertical velocity components of both objects are the same, and they reach the ground at the same time. However, the velocity of object ii also has a horizontal component.

Both objects reach the ground at the same time. Object ii strikes the ground with greater speed.

4. (8 points) A driver is driving a car down hill with a constant acceleration, and the position can be described with a motion diagram as shown. What is the direction of the car's acceleration?



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The displacements become progressively smaller as the car goes down the hill, so the velocity is becoming less and less "'downhill". The acceleration, therefore, is uphill.

5. (8 points) In which of these situations *must* an object's speed be increasing?

   I. The object has a positive acceleration.
   II. A component of the object's acceleration is in the same direction as its velocity.
   III. A component of the object's acceleration is perpendicular its velocity.

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The sign of an acceleration (situation I) depends on the choice of coordinate system, so a positive acceleration will **not** necessarily result in increasing speed.

The component of acceleration perpendicular to a velocity can only change the direction of the velocity (for example, a centripetal acceleration).

But if there's a component of acceleration in the same direction as a velocity, then the change in velocity is in the same direction as the velocity, and the speed will increase.

Only in situation II.

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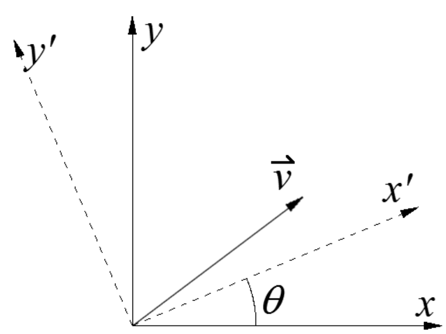
6. (8 points) A velocity vector $\vec{v}$ has components $v_x = 4.0\,\mathrm{m/s}$ and $v_y = 3.0\,\mathrm{m/s}$ in the $x{-}y$ coordinate system. What is the magnitude of this velocity vector in the $x'{-}y'$ coordinate system, which is rotated an angle $\theta = 25°$ as shown from the $x{-}y$ system?



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The magnitude in the original coordinate system is

$$|\vec{v}| = \sqrt{v_x^2 + v_y^2} = \sqrt{(4.0\,\mathrm{m/s})^2 + (3.0\,\mathrm{m/s})^2} = 5.0\,\mathrm{m/s}$$

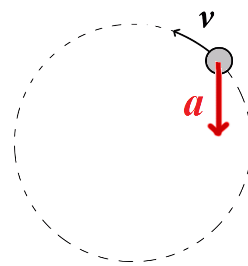
The vector will have different components in the rotated coordinate system, but the vector itself cannot change.

$5.0\,\mathrm{m/s}$

7. (8 points) A car is moving with a decreasing speed to around a circle with radius 10 m. The speed v of the car depends on time t according to

$$v(t) = 40\,\mathrm{m/s} - \left(10\,\mathrm{m/s^2}\right) t$$

At time $t = 3\,\mathrm{s}$ the car is at the location as shown. What is the direction of the acceleration of the car at this particular moment?



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As the car is moving in a circle, there must be a component of its acceleration toward the center (the centripetal acceleration). As the car's speed is decreasing, there must be a component of its acceleration opposite its velocity (the tangential acceleration).