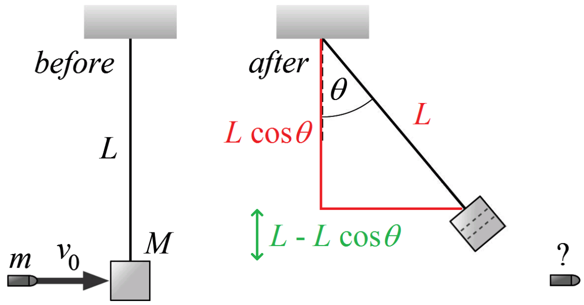

I. (16 points) A pendulum bob of mass M is hanging at rest from an ideal string of length L. A bullet of mass m traveling horizontally at speed v_0 strikes it and passes through. The bob loses no mass, and swings up to a maximum angle θ from the vertical, as shown. What is the speed of the bullet after it emerges from the bob? Express your answer in terms of parameters defined in the problem, and physical or mathematical constants. (*On Earth.*)

. . . .



Consider first the energy of a system consisting of the bob and the Earth after the bullet as passed through the bob. In the Energy Principle,

$$W_{\text{ext}} = \Delta K + \Delta U + \Delta E_{\text{th}}$$

no external forces do work (the tension in the string is perpendicular to the bob's displacement at every instant) so $W_{\text{ext}} = 0$. There are no internal dissipative forces, so $\Delta E_{\text{th}} = 0$. The only internal conservative force is that due to gravity, so $\Delta U = Mgh_f - Mgh_i$, where "f" and "i" denote final and initial states, respectively. The Earth's velocity change is negligible, so $\Delta K = \frac{1}{2}MV_f^2 - \frac{1}{2}MV_i^2$. Then

$$0 = \left(\tfrac{1}{2}MV_f^2 - \tfrac{1}{2}MV_i^2\right) + (Mgh_f - Mgh_i) + 0$$

If we *choose* $h_i = 0$ then $h_f = L\,(1 - \cos\theta)$. When the bob reaches its highest point, its speed is zero, so $V_f = 0$. Therefore,

$$0 = \left(0 - \tfrac{1}{2}MV_i^2\right) + (MgL\,(1 - \cos\theta) - 0)$$

Solve for V_i, the speed of the bob just after the bullet passes through it.

$$\tfrac{1}{2}MV_i^2 = MgL\,(1 - \cos\theta) \qquad \Rightarrow \qquad V_i = \sqrt{2gL\,(1 - \cos\theta)}$$

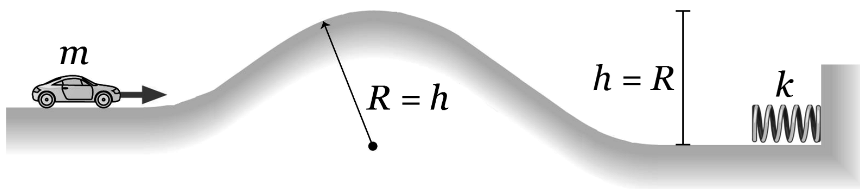
Consider next the momentum of a system consisting of the bob and the bullet while the bullet is passing through the bob. During this brief time, external forces are negligible in comparison to the internal forces between the bullet and the bob. Momentum, therefore, is conserved in this system during the interaction.

$$\Delta\vec{p} = 0 \qquad \Rightarrow \qquad \vec{p}_i = \vec{p}_f \qquad mv_i + MV_i' = mv_f + MV_f'$$

where the motion is in one dimension so signs will indicate direction. The bob is stationary before the bullet hits it, so $V_i = 0$. The velocity of the bob just after the bullet leaves it, is the velocity of the bob just before it swings up, so $V_f' = V_i$, from above.

$$mv_0 + 0 = mv_f + M\sqrt{2gL\,(1 - \cos\theta)} \qquad \Rightarrow \qquad v_f = v_0 - \frac{M}{m}\sqrt{2gL\,(1 - \cos\theta)}$$

II. (16 points) A toy car with a mass $m = 220\,\text{g}$ is given a push and released on a frictionless surface. It passes over a "hill" which has a radius $R = 12\,\text{cm}$ at the top, and which is $h = 12\,\text{cm}$ above the level ground on the other side, as shown. The car then strikes a spring with Hooke's Law constant $k = 64\,\text{N/m}$. If the car remains in contact with the ground at all times (*think about the implications for the speed of the car at the top of the hill!*), what is the maximum possible compression of the spring? (*On Earth.*)



. . . .

To achieve maximum possible spring compression, the car must crest the hill with the maximum possible kinetic energy (and so must have the maximum possible speed without losing contact with the hill). Apply Newton's Second Law to the car at the top of the hill. There's a normal force $\vec{n}$ upward, and a gravitational force $m\vec{g}$ downward. The acceleration of the car is downward, toward the center of the hill, so I'll choose the positive "c" axis in that direction. Newton's Second Law in the "c" direction is

$$\sum F_c = mg - n = ma_c = m\frac{v^2}{r}$$

but at its very fastest, the car is in contact with the hill but the hill isn't holding it up ($n = 0$). So

$$mg = m\frac{v^2}{r} \qquad \Rightarrow \qquad mv^2 = mgR \qquad \Rightarrow \qquad \tfrac{1}{2}mv^2 = mgR/2$$

Now consider a system consisting of the car, the Earth, and the spring. In the Energy Principle,

$$W_{\text{ext}} = \Delta K + \Delta U + \Delta E_{\text{th}} = \Delta K + \Delta U_g + \Delta U_s + \Delta E_{\text{th}}$$

no external forces do work (the normal force from the surface is perpendicular to the car's displacement at every instant) so $W_{\text{ext}} = 0$. There are no internal dissipative forces, so $\Delta E_{\text{th}} = 0$. There is an internal conservative force due to gravity, so $\Delta U_g = mgh_f - mgh_i$, where "$i$" and "$f$" denote initial and final states, at the top of the hill and at maximum compression, respectively. There is an internal conservative force due to the spring, so $\Delta U_s = \frac{1}{2}ks_f^2 - \frac{1}{2}ks_i^2$. The Earth's velocity change is negligible, so $\Delta K = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$.

$$0 = \left(\tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2\right) + (mgh_f - mgh_i) + \left(\tfrac{1}{2}ks_f^2 - \tfrac{1}{2}ks_i^2\right) + 0$$

The initial kinetic energy of the car was determined by Newton's Second Law above. At maximum compression the speed of the car is zero. If we *choose* the height of the car to be zero at maximum compression, then $h_f = 0$ and $h_i = R$. The spring's compression is zero before the car hits it.

$$0 = (0 - mgR/2) + (0 - mgR) + \left(\tfrac{1}{2}ks_f^2 - 0\right)$$

Solve for the final compression of the spring.

$$\tfrac{1}{2}ks_f^2 = 3mgR/2 \qquad \Rightarrow \qquad s_f = \sqrt{\frac{3mgR}{k}} = \sqrt{\frac{3\,(0.22\,\text{kg})\,(9.81\,\text{N/kg})\,(0.12\,\text{m})}{64\,\text{N/m}}} = 0.11\,\text{m}$$

1. (6 points) In the problem above, assuming the initial push on the car is sufficient for it to go over the hill and that it still remains in contact with the ground at all times, what is the minimum possible compression of the spring, Δs_{min}, that stops the car?

. . . .

To achieve minimum possible spring compression, the car must crest the hill with the minimum possible kinetic energy (and so must have a speed only infinitesimally greater than zero). The speed of the car is also zero when the spring compresses, so its kinetic energy change is zero. No work is done by external forces on a system consisting of the car, the Earth, and the spring. There are no thermal energy changes. The initial spring potential energy and the final gravitational potential energy can be chosen to be zero. Use the Energy Principle,

$$W_{\text{ext}} = \Delta K + \Delta U_g + \Delta U_s + \Delta E_{\text{th}} \qquad \Rightarrow \qquad 0 = 0 + (0 - mgh) + \left(\tfrac{1}{2}ks_f^2 - 0\right) + 0$$

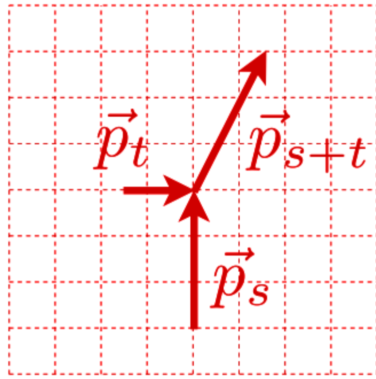
Solve for the final compression of the spring.

$$\tfrac{1}{2}ks_f^2 = mgh \qquad \Rightarrow \qquad s_f = \sqrt{2mgh/k}$$

Note that this question could also be answered by the process of elimination. The compression can't be zero if the car crests the hill. This situation is different from the problem above, so the compression can't be the same. $\sqrt{gR}$ has dimensions of speed, not distance. mg/k is the compression of the spring if it is vertical and the car is allowed to come to equilibrium on it. $\sqrt{2mgh/k}$ is the only remaining answer choice.

2. (6 points) A 525 kg great white shark, s, swimming due north at 9.5 m/s, ambushes a 235 kg tuna, t, swimming due east at 11 m/s, and swallows it whole. Which of the following represents the momentum vectors for this situation?

. . . .



At the instant the shark ambushes the tuna, forces external to the shark-tuna system are negligible compared to the forces between the shark and the tuna. Momentum in conserved in the shark-tuna system. The momentum after the ambush must be the same as the total (vector sum) momentum before the ambush.

$$\vec{p}_{s+t} = \vec{p}_s + \vec{p}_t$$

III. (16 points) In the above scenario, what is the *speed* of the shark (with belly full of tuna) immediately after the ambush?

. . . .

Since $\vec{p}_s$ and $\vec{p}_t$ the magnitude of $\vec{P}_{s+t}$ can be found using the Pythagorean Theorem.

$$p_{s+t} = \sqrt{p_s^2 + p_t^2}$$

But

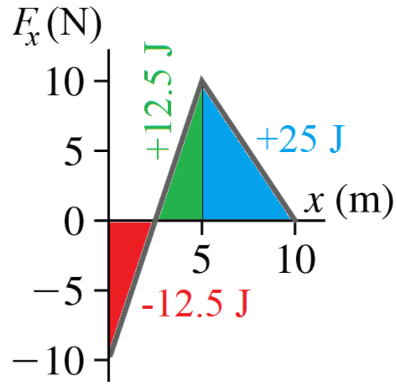
$$p_{s+t} = m_{s+t} v_{s+t} \qquad \Rightarrow \qquad v_{s+t} = \frac{p_{s+t}}{m_{s+t}} = \frac{\sqrt{p_s^2 + p_t^2}}{m_{s+t}}$$

So

$$v_{s+t} = \frac{\sqrt{(m_s v_s)^2 + (m_t v_t)^2}}{m_s + m_t} = \frac{\sqrt{\left[\,(525\,\text{kg})\,(9.5\,\text{m/s})\,\right]^2 + \left[\,(235\,\text{kg})\,(11\,\text{m/s})\,\right]^2}}{(525\,\text{kg}) + (235\,\text{kg})} = 7.4\,\text{m/s}$$

3. (8 points) A 2.0 kg object is traveling in the $+x$ direction at 7.0 m/s. When it arrives at the origin, it is subject to a varying force as shown. What is the speed of the object at $x = 10$ m?

. . . .



$$W = \int \vec{F} \cdot d\vec{s} = \Delta K = \tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2$$

the change in the object's kinetic energy is the area under the curve. From the graph, the area under the curve from the origin to $x = 10$ m is +25 J.

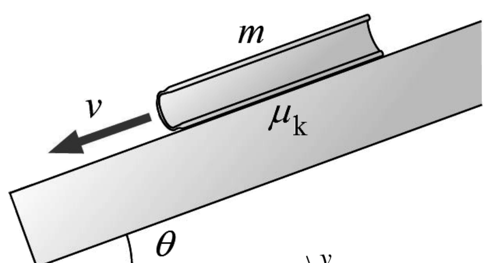
$$\tfrac{1}{2}mv_f^2 = \tfrac{1}{2}mv_i^2 + W$$

So

$$v_f = \sqrt{v_i^2 + \frac{2W}{m}} = \sqrt{(7.0\,\text{m/s})^2 + 2\left(\frac{25\,\text{J}}{2.0\,\text{kg}}\right)} = 8.6\,\text{m/s}$$

4. (8 points) A book of mass m slides down a slab that makes an angle θ with the horizontal. The coefficient of kinetic friction between the book and the slab is μ_k, so the book slides at constant speed v. At what *rate* is thermal energy increasing in the book-slab system? (*On Earth.*)

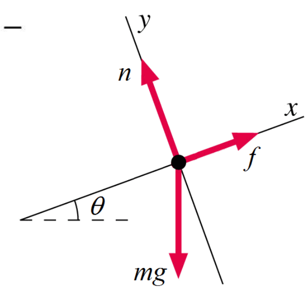
. . . .



The thermal energy change in the system is

$$\Delta E_{\text{th}} = fs$$

where $\vec{f}$ is the force of friction and s is the distance the book slides,. Sketch a Free Body Diagram and use Newton's Second Law to find



$$f = mg\sin\theta \qquad \text{so} \qquad \Delta E_{\text{th}} = mg\sin\theta\, s$$

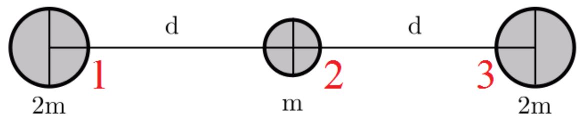
The rate at which thermal energy changes is power

$$P = \frac{d\,\Delta E_{\text{th}}}{dt} = \frac{d}{dt}\left[mg\sin\theta\, s\right] = mg\sin\theta\,\frac{ds}{dt} = mgv\sin\theta$$

5. (8 points) Consider three point masses of mass on a line. The central particle has mass m and two identical particles of mass $2m$ sit a distance d to the right and left of the central particle. With respect to zero at infinite separation, what is the universal gravitational potential energy in this system?

. . . .

The potential energy of the system is the sum of the potential energy of all the pairs of particles in the system. The particles have been numbered for reference.

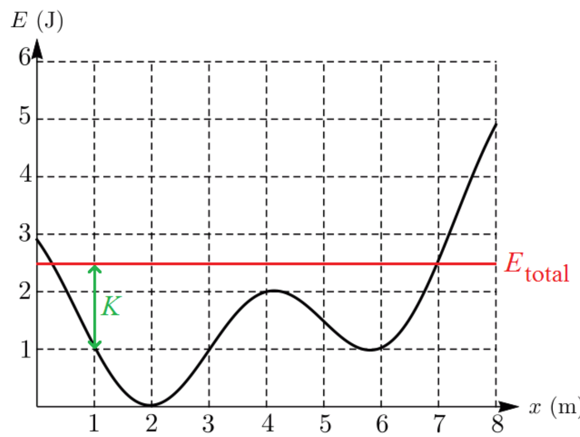


$$U = U_{12} + U_{13} + U_{23}$$

$$= \frac{-Gm_1m_2}{r_{12}} + \frac{-Gm_1m_3}{r_{13}} + \frac{-Gm_2m_3}{r_{23}} = \frac{-G\,2m\,m}{d} + \frac{-G\,2m\,2m}{2d} + \frac{-G\,m\,2m}{d} = \frac{-6Gm^2}{d}$$

6. (8 points) The graph shows the potential energy of a system as a function of the position, x, of a 3.0 kg particle within it. If the particle has a velocity of $\vec{v} = +1.0\,\hat{x}\,\text{m/s}$ at $x = 1.0\,\text{m}$, where, if anywhere, does the particle turn around?

. . . .



At $x = 1.0\,\text{m}$ the particle's kinetic energy is

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}\,(3.0\,\text{kg})\,(1.0\,\text{m/s})^2 = 1.5\,\text{J}$$

From the graph, the potential energy of the system is 1.0 J when the particle is at $x = 1.0\,\text{m}$, so the total energy of the system is

$$E_{\text{total}} = K + U = 1.5\,\text{J} + 1.0\,\text{J} = 2.5\,\text{J}$$

From the graph, the potential energy of the system is 2.5 J when the particle is at $x = 7\,\text{m}$, so the kinetic energy of the particle must be zero at that location. The particle stops and turns around at

$$x = 7\,\text{m}$$

7. (8 points) I push two balls across a frictionless floor with the same applied force for 10 m. Ball 1 has mass $m_1 = 1\,\text{kg}$, and ball 2 has mass $m_2 = 4\,\text{kg}$, so $m_2 = 4m_1$. Compare the resulting momentum magnitude of ball 2, p_2, with that of ball 1, p_1.

. . . .

Since the same force is applied to each ball for the same distance, the same work is done on each, and they'll have the same kinetic energy.

$$K = \tfrac{1}{2}mv^2 = \frac{(mv)^2}{2m} \qquad \Rightarrow \qquad p = \sqrt{2mK}$$

so the object with 4 times the mass with have $\sqrt{4}$ times the momentum.

$$p_2 = 2p_1$$