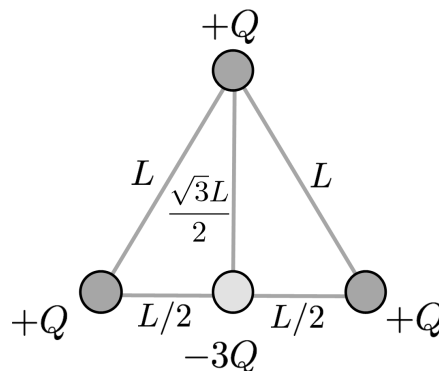


- I. (17 points) Three point charges, each carrying a charge $Q = +6.0 \text{ nC}$, are placed on an equilateral triangle of side length $L = 3.0 \text{ mm}$. An additional point charge, carrying a charge $-3Q$, is placed on one side of the triangle as drawn below. Calculate the electric potential energy of this system, relative to zero when the four charges are infinitely far away from each other.

The potential energy of a system of two charges, relative to zero for infinite separation, is

$$U = K \frac{q_1 q_2}{r}$$



The potential energy of the system shown, is the sum of the potential energies of all possible pairs of charges within it. Three pairs can be formed from the three charges on the vertices:

$$U_{\text{vertices}} = 3 \left(K \frac{Q Q}{L} \right) = 3K \frac{Q^2}{L}$$

The charge centered on the bottom side makes three pairs, one with each of the charges on a vertex. Note that this charge is a distance $L/2$ from two of the vertices, and a distance $L\sqrt{3}/2$ from the third.

$$U_{\text{side}} = 2 \left(K \frac{(-3Q) Q}{L/2} \right) + K \frac{(-3Q) Q}{L\sqrt{3}/2} = -12K \frac{Q^2}{L} - 6K \frac{Q^2}{L\sqrt{3}}$$

The potential energy of the system is

$$\begin{aligned} U_{\text{total}} &= U_{\text{vertices}} + U_{\text{side}} = 3K \frac{Q^2}{L} - 12K \frac{Q^2}{L} - 6K \frac{Q^2}{L\sqrt{3}} = \frac{KQ^2}{L} \left(-9 - \frac{6}{\sqrt{3}} \right) \\ &= \frac{(8.988 \times 10^9 \text{ N}\cdot\text{m}^2/\text{C}^2) (6.0 \times 10^{-9} \text{ C})^2}{3.0 \times 10^{-3} \text{ m}} \left(-9 - \frac{6}{\sqrt{3}} \right) = -1.3 \times 10^{-3} \text{ J} = \mathbf{-1.3 \text{ mJ}} \end{aligned}$$

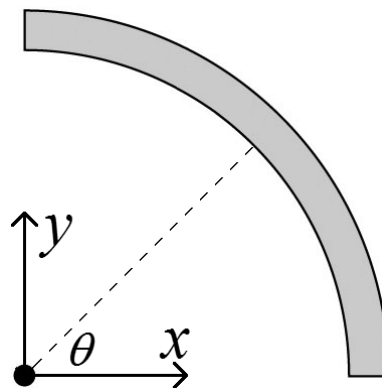
- II. (16 points) A thin non-uniform charged rod of length L is bent into a quarter-circle. When placed as shown, it has a linear charge density λ that varies with angle θ according to

$$\lambda = \lambda_0 \sin \theta$$

where λ_0 is a positive constant and θ is measured as usual, from the $+x$ axis toward the $+y$ axis. What is the magnitude of the resulting electric potential at the center of the arc (the origin), with respect to zero at infinite distance?

An element of charge dq , with an element of arc length ds , generates an element of potential dV . Since the rod is “thin”, this element of charge is point-like, so

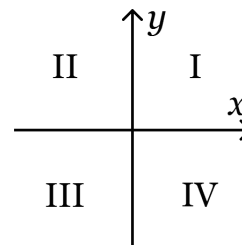
$$\begin{aligned} V &= \int dV = \int K \frac{dq}{r} = \int K \frac{\lambda ds}{r} = \int_0^{\pi/2} K \frac{\lambda_0 \sin \theta r d\theta}{r} = K \lambda_0 \int_0^{\pi/2} \sin \theta d\theta \\ &= K \lambda_0 \left[-\cos \theta \right]_0^{\pi/2} = -K \lambda_0 \left[\cos(\pi/2) - \cos(0) \right] = -K \lambda_0 [0 - 1] = K \lambda_0 \end{aligned}$$



1. (5 points) In the problem above, in what quadrant, if any, is the direction of the potential at the center of the arc (the origin)?

Potential is a scalar!

The potential has no direction.



- III. (17 points) Calculate the magnitude of the electric field *inside* a solid, non-uniformly charged sphere of radius R . The charge density inside the sphere is

$$\rho(r) = \rho_0 \frac{r^2}{R^2}$$

for $r < R$, where ρ_0 is a constant and r is the distance from the center of the sphere.

Use Gauss' Law, $\epsilon_0 \Phi = q_{\text{in}}$.

To find the flux, choose a Gaussian Surface with the symmetry of the charge distribution, and that passes through the point at which the field is to be found. This is a sphere of radius r .

$$\Phi = \int \vec{E} \cdot d\vec{A} = \int E \cos \theta dA$$

On that Gaussian Surface, the electric field has a uniform magnitude E at every point. The electric field vector is perpendicular to the surface, and so parallel to the vector representing the element of area, at every point.

$$\Phi = \int E \cos 0 dA = E \int dA = EA = E 4\pi r^2$$

The charge inside this Gaussian Surface can be determined from the volume charge density.

$$\rho = \frac{dq}{dV} \quad \Rightarrow \quad q_{\text{in}} = \int \rho dV$$

Since the volume charge density varies with radius, choose volume elements that are “small” in the radial direction. There are thin spherical shells of radius r and thickness dr . Add up (integrate) the charge in all the shells from the center to the radius of Gaussian Surface, r .

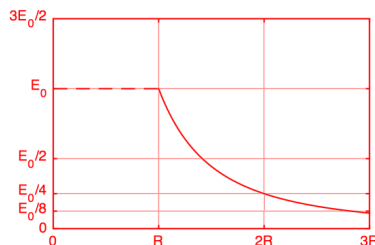
$$q_{\text{in}} = \int_0^r \rho_0 \frac{r^2}{R^2} 4\pi r^2 dr = \frac{4\pi\rho_0}{R^2} \int_0^r r^4 dr = \frac{4\pi\rho_0}{R^2} \frac{r^5}{5} \Big|_0^r = \frac{4\pi\rho_0}{5R^2} r^5$$

Relate the flux to the charge with Gauss' Law, and solve for the field magnitude.

$$\epsilon_0 E 4\pi r^2 = \frac{4\pi\rho_0}{5R^2} r^5 \quad \Rightarrow \quad E = \frac{\rho_0 r^3}{5\epsilon_0 R^2}$$

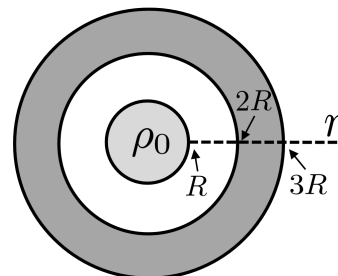
2. (5 points) Which sketch shows the electric field *outside* the sphere of the previous question? (The electric field inside the sphere is *not* shown accurately. It is rendered as a straight dashed line).

Outside a spherically symmetric charge distribution, the field must be the same as the field that would be produced if the entire charge were concentrated at a point in the center. The field due to a point charge follows an inverse square law with distance. So, the field at $2R$ must have one-fourth the magnitude of the field at the surface.



3. (5 points) An insulating spherical bead of radius R is charged with an uniform positive volume charge density ρ_0 . The bead is placed in the center of a hollow conducting sphere of inner radius $2R$ and outer radius $3R$. The hollow conducting sphere is initially neutral. What is surface charge density η on the exterior surface of the hollow sphere?

Start by considering a Gaussian Surface entirely within the thickness of the hollow conducting sphere. The field in the conducting material is zero at equilibrium, so the flux through the Gaussian Surface is zero, so the charge within the Gaussian Surface is zero. But we know there is a charge



$$Q_{\text{bead}} = \rho V = \rho_0 \frac{4}{3} \pi R^3$$

on the inner bead. Therefore, there must be the opposite charge

$$Q_{\text{inner}} = \rho V = -\rho_0 \frac{4}{3} \pi R^3$$

on the inner surface of the conducting sphere. Since this conducting sphere is neutral, there must be the opposite charge

$$Q_{\text{outer}} = \rho V = \rho_0 \frac{4}{3} \pi R^3$$

on the outer surface of the conducting sphere. The area charge density on this outer surface is

$$\eta = \frac{Q_{\text{outer}}}{A} = \frac{\rho_0 (4/3) \pi R^3}{4\pi (3R)^2} = \frac{\rho_0 R}{27}$$

4. (5 points) A proton is sent with a velocity $v_0 = 3.0 \times 10^6$ m/s (1% of the speed of light) toward a mercury nucleus containing $Z = 80$ protons, initially located infinitely far away. The radius of a mercury nucleus is so small ($10 \text{ fm} = 10^{-14} \text{ m}$), that the nucleus can be described as a point charge. How close to the nucleus will the proton get? Reminder: 1 pico-meter (pm) = 10^{-12} m .

Use the Work-Energy Theorem. Choose a system consisting of the proton and the mercury nucleus. No external forces do work on this system, and there are no non-conservative forces within it to change its thermal energy.

$$W_{\text{ext}} = \Delta K + \Delta U + \Delta E_{\text{th}} \quad \Rightarrow \quad 0 = \Delta K + \Delta U + 0$$

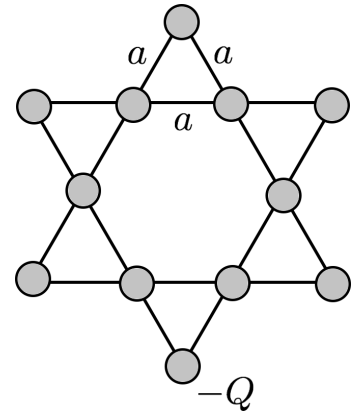
The mass of a mercury nucleus is so much greater than that of a proton, that the mercury nucleus can be considered fixed in place, so the only kinetic energy change in the system is due to the proton. The potential energy change in the system is due to the internal electric force.

$$0 = \left(\frac{1}{2} m_p v_f^2 - \frac{1}{2} m_p v_0^2 \right) + \left(K \frac{Q_p Q_{\text{Hg}}}{r_f} - K \frac{Q_p Q_{\text{Hg}}}{r_i} \right) = \left(0 - \frac{1}{2} m_p v_0^2 \right) + \left(K \frac{e(Ze)}{r_f} - 0 \right)$$

where v_f , the speed of the proton at the instant of closest approach, is zero. As the proton starts infinitely far away, the initial potential energy of the system is zero, too. Solve for r_f .

$$r_f = \frac{2KZe^2}{m_p v_0^2} = \frac{2(8.988 \times 10^9 \text{ N}\cdot\text{m}^2/\text{C}^2)(80)(1.602 \times 10^{-19} \text{ C})^2}{(1.673 \times 10^{-27} \text{ kg})(3.0 \times 10^6 \text{ m/s})^2} = 2.45 \text{ pm}$$

5. (5 points) Twelve identical point charges, each carrying a charge $-Q$, are organized on the “kagome” pattern sketched below. Each triangle in the pattern is equilateral with a side length a . The electric potential energy of the system is U_0 . What is the electric potential energy U_1 of the system when the distance a is multiplied by 3? (“Kagome” is the Japanese word for traditional bamboo baskets having this geometry.)



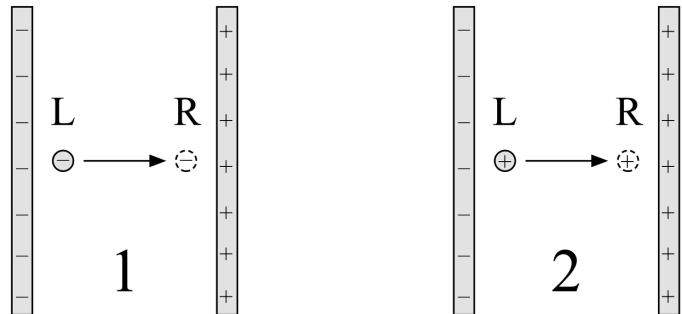
Since the potential energy of each pair of charges in the system is inversely proportional to the distance between them, the potential energy of the entire system must be inversely proportional to a .

$$U_1 = U_0/3$$

6. (5 points) Consider two scenarios inside identical capacitors. In each scenario, the capacitor is charged identically, with charge $-Q$ on the left-hand plate and $+Q$ on the right-hand plate. In scenario 1, a negatively charged particle is moved from the left (L) to the right (R). In scenario 2, a positively charged particle (of equal magnitude) is moved from L to R. In each case, the potential difference is defined as $\Delta V = V_R - V_L$, and the potential energy difference is defined as $\Delta U = U_R - U_L$. Which of the statements below is true?

In situation 2, the particle would not move spontaneously from left to right. The potential energy of this system increases when the particle does this ($\Delta U_2 > 0$). Since the potential change is the potential energy change per unit charge, and the particle in situation 2 is positive, this particle is moving through a positive potential difference ($\Delta V_2 > 0$).

Since the capacitors in the two situations are charged identically, and potential differences do not depend on the nature of the probe charge used to determine them, the potential change in situation 1 must be the same as that in situation 2 ($\Delta V_1 > 0$). However, the particle in situation 1 would move spontaneously from left to right. The potential energy of this system decreases when the particle does this ($\Delta U_1 < 0$).

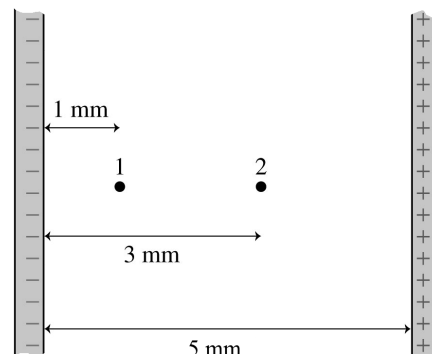


$$\Delta V_1 > 0; \Delta U_1 < 0; \Delta V_2 > 0; \Delta U_2 > 0$$

7. (5 points) The plates in an ideal parallel plate capacitor are 5.0 mm apart, as shown. The potential difference between these plates is 15 V. Point 1 is 1.0 mm from the negative plate, and point 2 is 3.0 mm from the negative plate. If it can be determined, what is the potential difference **from** point 1 **to** point 2?

As it would require external work to force a positive test charge from point 1 to point 2, the potential difference from point 1 to point 2 must be positive. As the potential difference within a parallel-plate capacitor depends linearly on distance, this potential difference must be

$$\Delta V_{1 \rightarrow 2} = \Delta V_{\text{total}} \frac{\Delta s_{1 \rightarrow 2}}{\Delta s_{\text{total}}} = (15 \text{ V}) \frac{2.0 \text{ mm}}{5.0 \text{ mm}} = +6.0 \text{ V}$$



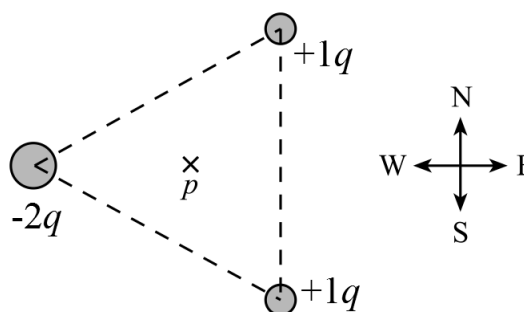
8. (5 points) Two positive charges $+q$ and a negative charge $-2q$ are placed at the vertices of an equilateral triangle, as shown. Use the convention that $V = 0$ at infinity.

Which statement about the point p , at the center of the triangle, is true?

Each charged particle is the same distance r from the point p . The $-2q$ charge contributes a potential of $-2Kq/r$ at the point p , and each of the q charges contribute $+Kq/r$. The potential at the point p must be zero.

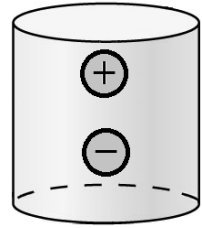
If a positive probe charge were placed at the point p , there would be an attractive force toward the $-2q$ charge, and a repulsive force from the $+q$ charges. The force, and thus the field, would be toward the West.

$$V = 0; \vec{E} \text{ points West.}$$



9. (5 points) A positively charged particle and a negatively charged particle, each having charge magnitude q , lie on the axis of a cylindrical surface, equidistant from the ends, as shown. Rank the flux through the top, side, and bottom of the cylinder, from greatest to least. Remember that electric flux can be positive or negative.

Field lines start on positive charges and end on negative charges. Because the positive charge is nearer the top surface than the negative charge is, the net flux through the top surface is outward. Because the negative charge is nearer the bottom surface than the positive charge is, the net flux through the bottom surface is outward. The area vectors for the cylinder surfaces point outward, so the angle between the field vectors and area vectors is between 0° and 90° on the top surface, and between 90° and 180° on the bottom surface. As flux is



$$\Phi_E = \int \vec{E} \cdot d\vec{A} = \int E \cos \theta dA$$

the flux through the top surface is positive, while that through the bottom surface is negative.

Because the charged particles are symmetrically placed, any field line that passes outward through the side of the cylinder as it heads away from the positive charge, must also pass inward through the side of the cylinder as it heads toward the negative charge. The flux through the side is zero.

$$\Phi_{\text{top}} > \Phi_{\text{side}} > \Phi_{\text{bottom}}$$

10. (5 points) A positively charged particle lies in the plane of a cylindrical surface's top, a distance d from the axis, as shown. The cylinder has height h . What is the sign, if any, of the flux through the cylinder's curved side?

Since the field from the charged particle is parallel to the top surface of the cylinder (perpendicular to the area vector), the flux through the top is zero.

At the bottom surface, field only exits the cylinder. The flux through the bottom is positive.

As there is no charge within the cylinder, the net flux through it must be zero. Therefore,

$$\Phi_{\text{side}} \text{ is negative.}$$

