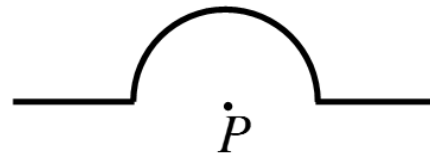


- I. (17 points) The magnetic field at point P due to a current through the wire is $5.0\ \mu\text{T}$ into the page. The curved portion of the wire is a semicircle of radius 2.0 cm , and the two straight segments are each 3.0 cm . Calculate the magnitude of the current. *Show your work, starting from a fundamental physical principle.* In what direction does it flow?

Use the Biot-Savart Law.

$$d\vec{B} = \frac{\mu_0 I}{4\pi} \frac{d\vec{\ell} \times \hat{r}}{|\vec{r}|^2} \quad \Rightarrow \quad dB = \frac{\mu_0 I}{4\pi} \frac{d\ell \sin \phi}{r^2}$$



In each of the straight segments, $d\vec{\ell}$ is parallel or anti-parallel to $\hat{r}$ ($\phi = 0$ or π) so $\sin \phi = 0$. These segments make no contribution to the magnetic field at point P .

At every location in the curved portion, $d\vec{\ell}$ is perpendicular to $\hat{r}$ ($\phi = \pi/2$) so $\sin \phi = 1$. Every location in the curved portion is also the same distance r from the point P , and the current through every element of length $d\vec{\ell}$ produces magnetic field in the same direction. The magnitude of the field is

$$B = \int dB = \int \frac{\mu_0 I}{4\pi} \frac{d\ell (1)}{r^2} = \frac{\mu_0 I}{4\pi r^2} \int_0^{\pi r} d\ell = \frac{\mu_0 I}{4\pi r^2} \ell \Big|_0^{\pi r} = \frac{\mu_0 I}{4\pi r^2} (\pi r - 0) = \frac{\mu_0 I}{4r}$$

Solve for I .

$$I = \frac{4rB}{\mu_0} = \frac{4(0.020\text{ m})(5.0 \times 10^{-6}\text{ T})}{4\pi \times 10^{-7}\text{ T}\cdot\text{m/A}} = 0.32\text{ A}$$

Consider the element of at the very “top” of the semicircle. The direction of $\hat{r}$ from that point to P is down the page. With the magnetic field at point P being into the page, the Right-Hand-Rule shows that the current must be flowing

to the right, or clockwise.

- II. (17 points) A long, straight conducting wire of radius R has a nonuniform current density $\vec{J}(r)$ whose magnitude depends on the distance r from the center according to

$$J(r) = J_0 \left[1 - \frac{r^3}{R^3} \right]$$

where J_0 is a positive constant. Find an expression for the magnetic field magnitude *inside* the wire, at radius $r < R$, in terms of parameters defined in the problem and physical or mathematical constants.

Use Ampere's Law.

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{\text{thru}} = \mu_0 \int \vec{J} \cdot d\vec{A}$$

From symmetry, the field lines must be circular. To find the field at an arbitrary distance $r < R$ from the center of the wire, choose an Amperian Loop that follows a field line of radius r . Then $\vec{B}$ is parallel to $d\vec{s}$ and has the same magnitude B at every point on the loop. Only current at a distance r or less from the center of the wire passes through this loop. Since the current varies with r , the area element dA should be "small" in this dimension. Choose dA to be a thin ring, with area $2\pi r dr$.

$$\begin{aligned} B \oint ds &= \mu_0 \int_0^r \vec{J}_0 \left[1 - \frac{r^3}{R^3} \right] 2\pi r dr \quad \Rightarrow \quad B 2\pi r = \mu_0 J_0 2\pi \left[\int_0^r r dr - \int_0^r \frac{r^4}{R^3} dr \right] \\ &= \mu_0 J_0 2\pi \left[\frac{r^2}{2} - \frac{r^5}{5R^3} \right]_0^r = \mu_0 J_0 2\pi \left[\frac{r^2}{2} - \frac{r^5}{5R^3} \right] \end{aligned}$$

Solve for B .

$$B = \mu_0 J_0 \left[\frac{r}{2} - \frac{r^4}{5R^3} \right]$$

1. (5 points) Outside the wire in the problem above ($r > R$), how does the magnitude of the magnetic field B depend on the distance r from the center of the wire?

Use Ampere's Law.

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{\text{thru}}$$

From symmetry, the field lines must be circular. Choose an Amperian Loop that follows a circular field line of radius r . Then $\vec{B}$ is parallel to $d\vec{s}$ and has the same magnitude B at every point on the loop.

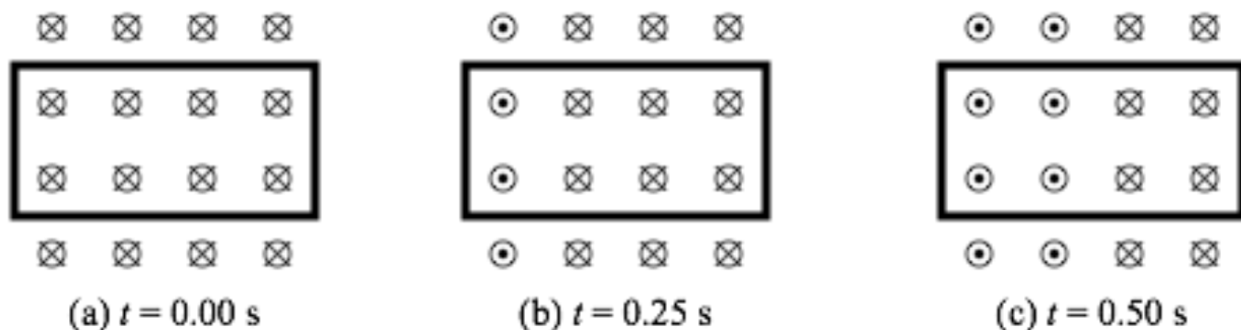
$$\oint \vec{B} \cdot d\vec{s} = B 2\pi r = \mu_0 I_{\text{thru}} \quad \Rightarrow \quad B = \frac{\mu_0 I_{\text{thru}}}{2\pi r}$$

So

Outside the wire, B is proportional to $1/r$.

- III. (16 points) A $12\text{ cm} \times 25\text{ cm}$ rectangular loop, which has a total resistance of $1.0\text{ m}\Omega$, is placed in a uniform 1.0 mT magnetic field, as shown in Figure (a). The applied field gradually flips direction — from into the page to out of the page — with the change sweeping slowly from left to right. The time elapsed between figures (a) and (c) is 0.50 s .

What is the induced current in the loop at time $t = 0.50\text{ s}$, when the field flip is halfway along the loop?



Use Faraday's Law to find the magnitude of the emf induced in the loop.

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt} \left[\int \vec{B} \cdot d\vec{A} \right] \quad \Rightarrow \quad |\mathcal{E}| = \left| \frac{d\Phi_B}{dt} \right| = \left| \frac{\Delta\Phi_B}{\Delta t} \right| = \left| \frac{\Phi_{Bf} - \Phi_{Bi}}{\Delta t} \right|$$

Note that the final flux through the loop Φ_{Bf} is zero at time $t = 0.50\text{ s}$, while the magnitude of the initial flux through the loop Φ_{Bi} at time $t = 0.50\text{ s}$ is just BA .

$$|\mathcal{E}| = \left| \frac{0 - BA}{\Delta t} \right| = \frac{BA}{\Delta t}$$

The current can be found from the definition of resistance.

$$\Delta V = IR \quad \Rightarrow \quad I = \frac{\Delta V}{R} = \frac{\mathcal{E}}{R} = \frac{BA}{R\Delta t} = \frac{(1.0 \times 10^{-3}\text{ T})(0.12\text{ m} \times 0.25\text{ m})}{(1.0 \times 10^{-3}\text{ }\Omega)(0.50\text{ s})} = \mathbf{0.060\text{ A}}$$

2. (5 points) What is the direction of the current, if any, in the loop in figure (c), as seen from above (the vantage point of the figures):

Use Lenz' Law. The flux through the loop is initially into the page. It decreases with time. The induced current must flow to create an induced field that opposes this change. Therefore, the induced field must be into the page. Using the Right-Hand-Rule, the current must be

Clockwise.

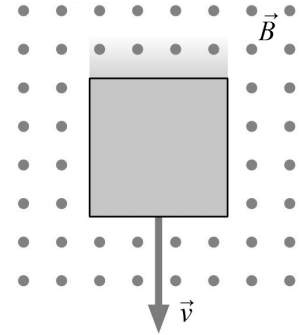
3. (5 points) The conducting slab moves down the page with velocity $\vec{v}$, through a magnetic field $\vec{B}$ directed out of the page. At what location, if any, in the conductor is the electric potential highest?

Consider a positively-charged particle free to move within the conductor. It moves downward with the slab, so the magnetic force

$$\vec{F} = q\vec{v} \times \vec{B}$$

is to the left. The left edge of the conductor becomes positively charged, leaving the right edge to be negatively charged. Electric field points from positive to negative charges, or rightward. Electric field also points from high to low potential, so

The electric potential is highest on the left edge of the conductor.



4. (5 points) Three identical conducting loops enter a magnetic field at identical velocities $\vec{v}$. At the instant shown, rank the current magnitudes in the loops from greatest to least.

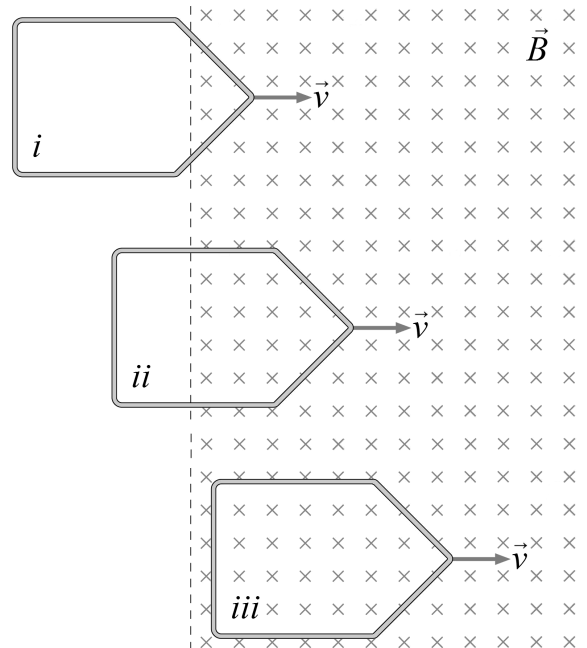
The current is proportional to the induced emf, $I = \mathcal{E}/R$, where R is the resistance of the loops.

Using Faraday's Law, the emf is related to the time rate of magnetic flux change through a loop.

$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$

In a time Δt , each loop advances a distance $v \Delta t$. That increases the area of field within loops i and ii by their height at the edge of the field times $v \Delta t$. This is greater for loop ii than for loop i . But as loop iii is entirely within the field, there is no flux change as it advances.

$$I_{ii} > I_i > I_{iii}$$



5. (5 points) Three long wires are aligned parallel to one another, with a uniform spacing d between wires. Each wire carries the current indicated in the figure. Rank order the magnitudes of the net force exerted on each wire.

The force on a current-carrying wire $\vec{F} = I\vec{\ell} \times \vec{B}$ depends on the current I in the wire and the field $\vec{B}$ it is in. From Ampere's Law, the field from a current-carrying wire depends on the current in the wire and the distance r between them. So the force magnitude between two parallel current-carrying wires will depend on each of the currents, and the distance between them, $F \propto I_1 I_2 / r$.

Consider the force F_L on the left wire. The center and right wires, having currents in opposite directions, produce field (and therefore force) in opposite directions.

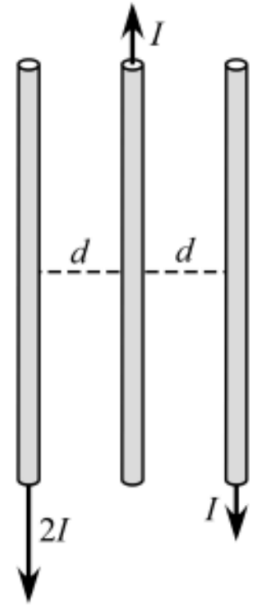
$$F_L \propto \frac{I_L I_C}{r_{LC}} - \frac{I_L I_R}{r_{LR}} = \frac{2I^2}{d} - \frac{2I^2}{2d} = \frac{I^2}{d}$$

Consider the force F_C on the center wire. The left and right wires, being on opposite sides, produce field (and therefore force) in opposite directions.

$$F_C \propto \frac{I_C I_L}{r_{CL}} - \frac{I_C I_R}{r_{CR}} = \frac{2I^2}{d} - \frac{I^2}{d} = \frac{I^2}{d}$$

Consider the force F_R on the right wire. The center and left wires, having currents in opposite directions, produce field (and therefore force) in opposite directions.

$$F_R \propto \frac{I_R I_C}{r_{RC}} - \frac{I_R I_L}{r_{RL}} = \frac{I^2}{d} - \frac{2I^2}{2d} = 0 \quad \text{so} \quad F_{\text{left}} = F_{\text{center}} > F_{\text{right}}$$

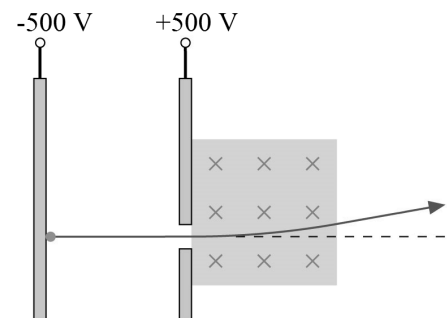


6. (6 points) Consider a particle that is released from rest at the -500 V plate of a capacitor. It accelerates toward the $+500\text{ V}$ plate, and passes through a small hole. Entering a uniform magnetic field directed into the page, its path is deflected up the page. Under what circumstances, if any, is this possible? (Neglect the effects of gravity.)

Electric field points from high potential to low potential. If the particle accelerates toward the high-potential plate, the force $\vec{F}_E = q\vec{E}$ on it must be in the direction of the electric field, so the particle must be negatively charged.

If a negatively-charged particle enters the magnetic field as shown, the force $\vec{F}_B = q\vec{v} \times \vec{B}$ would be down the page. So,

This is not possible.



7. (5 points) Initially, an inductor stores magnetic energy U_0 with a current I_0 running through it. Starting at time $t = 0$, the current changes linearly, as indicated by the graph. What is the final energy stored in the inductor?

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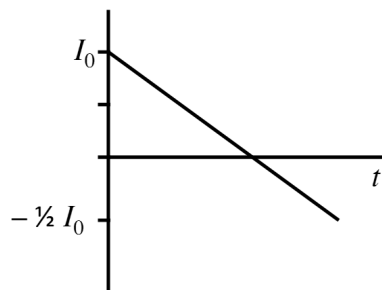
The energy stored in an inductor is

$$U_L = \frac{1}{2}LI^2$$

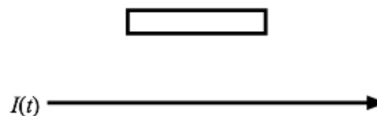
At the final time, the current has half the magnitude it did at $t = 0$.

Therefore, the energy is

$$U_0/4$$



8. (5 points) A rectangular conducting loop is positioned above a long wire that carries a current $I(t)$. If $I(t)$ is increasing with time, what force or torque does the loop experience?



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From the Biot-Savart Law and the Right-Hand-Rule (or a short cut) the field from the wire through the loop is out of the page. As the current in the wire increases, the flux out of the page through the loop increases. From Lenz' Law, current must flow in the loop to create flux opposing that change, or into the page. Using the Right-Hand-Rule again, the current in the loop must flow clockwise.

The force on each side of the loop can be considered, using $\vec{F} = I\vec{L} \times \vec{B}$, where $\vec{B}$ is the field due to the wire. The force on the left side of the loop is to the right, and the force on the right side of the loop is to the left. Both are in the plane of the page and have the same magnitude, producing no net torque or force. The force on the upper side of the loop is down the page, and the force on the lower side of the loop is up the page. Both are in the plane of the page, producing no net torque. The upper side of the loop, however, is farther from the wire, and so is in a weaker field. The magnitude of the force on the upper side, then, is less than that on the lower side. Therefore, there is

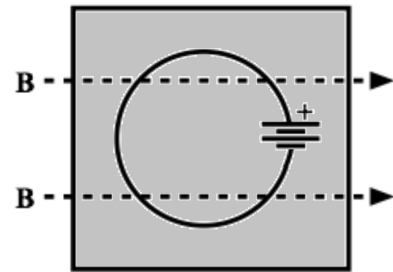
Only a force up the page.

9. (5 points) A circular loop of current rests on a table top. A large, uniform magnetic field is applied parallel to the table top, oriented from left to right, as shown. How does the loop respond?

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The battery will produce a counter-clockwise current in the loop. The magnetic moment $\vec{\mu}$ of the loop, then, will be out of the page. Since the torque on current loop in a magnetic field is $\vec{\tau} = \vec{\mu} \times \vec{B}$, the Right-Hand-Rule indicates the torque is directed up the page. With the torque in that direction,

Its left edge lifts off the table.



10. (5 points) A negatively charged particle, moving with velocity $\vec{v}$ as shown, experiences a magnetic Lorentz force oriented directly out of the page. What is the direction of the magnetic field at the particle's location?

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The magnetic force on a charged particle is $\vec{F} = q\vec{v} \times \vec{B}$. Using the Right-Hand-Rule for the vector product, the magnetic field must be in this direction:

