

I. (16 points) The electric potential in a certain region of space depends on position according to

$$V(x, y, z) = (2.0 \text{ V/m}^2) (x^2 + y^2) + (3.0 \text{ V}) \sin\left(\frac{z}{4.0 \text{ m}}\right) + 6.0 \text{ V}$$

Find an algebraic and/or trigonometric expression for the electric field in that same region of space.

Electric field is related to electric potential by

$$E_s = -\frac{\delta V}{\delta s}$$

So

$$\begin{aligned} E_x &= -\frac{\delta V}{\delta x} = -\frac{\delta}{\delta x} \left[(2.0 \text{ V/m}^2) (x^2 + y^2) + (3.0 \text{ V}) \sin\left(\frac{z}{4.0 \text{ m}}\right) + 6.0 \text{ V} \right] = -[(2.0 \text{ V/m}^2) 2x] \\ &= -(4.0 \text{ V/m}^2) x \end{aligned}$$

$$\begin{aligned} E_y &= -\frac{\delta V}{\delta y} = -\frac{\delta}{\delta y} \left[(2.0 \text{ V/m}^2) (x^2 + y^2) + (3.0 \text{ V}) \sin\left(\frac{z}{4.0 \text{ m}}\right) + 6.0 \text{ V} \right] = -[(2.0 \text{ V/m}^2) 2y] \\ &= -(4.0 \text{ V/m}^2) y \end{aligned}$$

$$\begin{aligned} E_z &= -\frac{\delta V}{\delta z} = -\frac{\delta}{\delta z} \left[(2.0 \text{ V/m}^2) (x^2 + y^2) + (3.0 \text{ V}) \sin\left(\frac{z}{4.0 \text{ m}}\right) + 6.0 \text{ V} \right] \\ &= -\left[(3.0 \text{ V}) \cos\left(\frac{z}{4.0 \text{ m}}\right) \left(\frac{1}{4.0 \text{ m}}\right) \right] = -(0.75 \text{ V/m}) \cos\left(\frac{z}{4.0 \text{ m}}\right) \end{aligned}$$

Therefore

$$\vec{E} = E_x \hat{i} + E_y \hat{j} + E_z \hat{k} = -(4.0 \text{ V/m}^2) x \hat{i} - (4.0 \text{ V/m}^2) y \hat{j} - (0.75 \text{ V/m}) \cos\left(\frac{z}{4.0 \text{ m}}\right) \hat{k}$$

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- II. (17 points) The starter motor of a car engine draws a current of 220 A from the battery. The copper wire to the motor has a conduction-electron density of 8.5×10^{28} per cubic meter. It is 3.8 mm in diameter and 1.5 m long. The starter motor runs for 1.2 s until the car engine starts. How far does an electron travel along the wire while the starter motor is on?

The drift speed v_d of the electrons can be determined from

$$I = nA v_d e \quad \Rightarrow \quad v_d = \frac{I}{nAe} = \frac{I}{n(\pi/4) D^2 e}$$

where D is the diameter of the wire. From constant-acceleration kinematics

$$x_f = x_i + v_i \Delta t + \frac{1}{2} a (\Delta t)^2 \quad \Rightarrow \quad \Delta x = v_i \Delta t$$

where the initial velocity of the electrons has magnitude v_d , and the acceleration a of the electrons is zero as the time required for the electrons to reach v_d is negligible compared to 1.2 s.

$$\Delta x = \left(\frac{I}{n(\pi/4) D^2 e} \right) \Delta t = \frac{4(220 \text{ A})(1.2 \text{ s})}{(8.5 \times 10^{28} \text{ m}^{-3}) \pi (3.8 \times 10^{-3} \text{ m})^2 (1.602 \times 10^{-19} \text{ C})} = 1.7 \times 10^{-3} \text{ m} = 1.7 \text{ mm}$$

1. (5 points) If, in the problem above, the 220 A flowed through an iron wire rather than a copper wire, how would the distance traveled by an electron compare? Iron has a conduction-electron density about twice that of copper.

The drift speed of an electron in a material is inversely proportional to the conduction-electron density

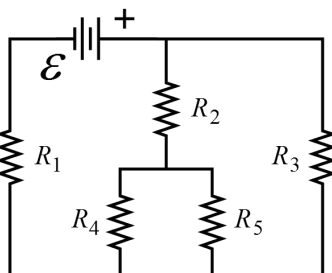
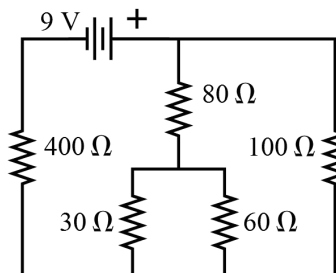
$$I = nA v_d e \quad \Rightarrow \quad v_d = \frac{I}{nAe}$$

so doubling the conduction-electron density would halve the drift speed. In the same time,

An electron would travel half as far in an iron wire as in a copper wire.

III. (17 points) In the circuit shown, what is the current through the $400\ \Omega$ resistor?

Re-draw the circuit “neater” and assign symbols to the emf and resistances.



Find the equivalent resistance of the circuit. Resistors R_4 and R_5 are in parallel.

$$R_{45} = \left(\frac{1}{R_4} + \frac{1}{R_5} \right)^{-1} = \left(\frac{1}{30\ \Omega} + \frac{1}{60\ \Omega} \right)^{-1} = 20\ \Omega$$

Resistors R_2 and R_{45} are in series.

$$R_{245} = R_2 + R_{45} = 80\ \Omega + 20\ \Omega = 100\ \Omega$$

Resistors R_{245} and R_3 are in parallel.

$$R_{2345} = \left(\frac{1}{R_{245}} + \frac{1}{R_3} \right)^{-1} = \left(\frac{1}{100\ \Omega} + \frac{1}{100\ \Omega} \right)^{-1} = 50\ \Omega$$

Resistors R_1 and R_{2345} are in series.

$$R_{12345} = R_1 + R_{2345} = 400\ \Omega + 50\ \Omega = 450\ \Omega$$

From the definition of resistance, the current supplied by the emf is

$$\Delta V = IR \Rightarrow I_{12345} = \frac{\Delta V_{12345}}{R_{12345}} = \frac{\mathcal{E}}{R_{12345}} = \frac{9\ \text{V}}{450\ \Omega} = 0.02\ \text{A}$$

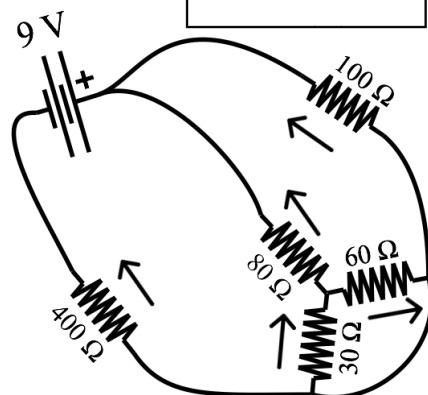
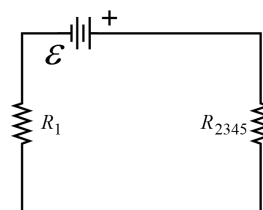
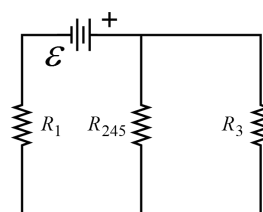
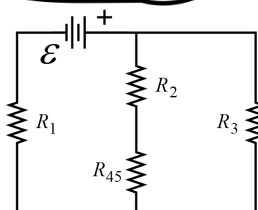
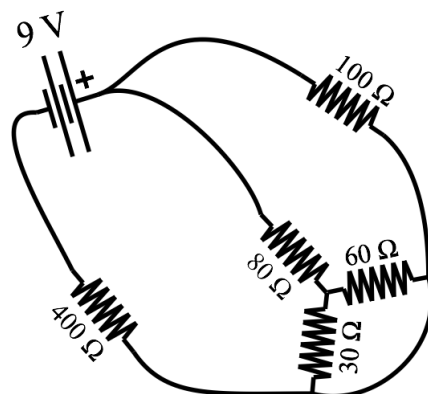
Since resistors R_1 and R_{2345} are in series, the current through R_1 , the $400\ \Omega$ resistor, is the same as the current through R_{12345} , or

20 mA

2. (5 points) If positive current directions are defined by the arrows as shown, and current through the $400\ \Omega$ resistor is I_{400} , etc., which equation is a valid expression of Kirchhoff's Loop Law?

Kirchhoff's Loop Law is that the sum of potential differences around a closed loop is zero. Remember that current flows from high to low potential through a resistor. Therefore, when following such a closed loop, traversing a resistor in the direction of current flow decreases the potential, while traversing it opposite the direction of current flow increases the potential. The only offered choice consistent with this is:

$$+9\ \text{V} + I_{100}(100\ \Omega) + I_{60}(60\ \Omega) + I_{30}(30\ \Omega) - I_{400}(400\ \Omega) = 0$$



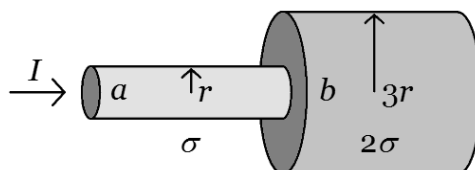
3. (5 points) A current I flows to the right through a cylindrical wire made of two segments, as shown. First, it flows through segment a , with radius r and conductivity σ . It continues on to flow through segment b , with radius $3r$ and conductivity 2σ . What is the ratio of the electric field magnitude in segment a to that in segment b ?

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The current is the same in each of the two segments.

$$I_a = I_b \Rightarrow J_a A_a = J_b A_b \Rightarrow \sigma_a E_a \pi R_a^2 = \sigma_b E_b \pi R_b^2$$

$$\frac{E_a}{E_b} = \frac{\sigma_b R_b^2}{\sigma_a R_a^2} = \frac{2\sigma (3r)^2}{\sigma r^2} = 18$$



4. (5 points) When a 6 V battery is connected across a solid conducting slab as shown on the left diagram, a current of 3.0 A flows through the circuit. What is the current if instead the battery is connected as shown on the right diagram?

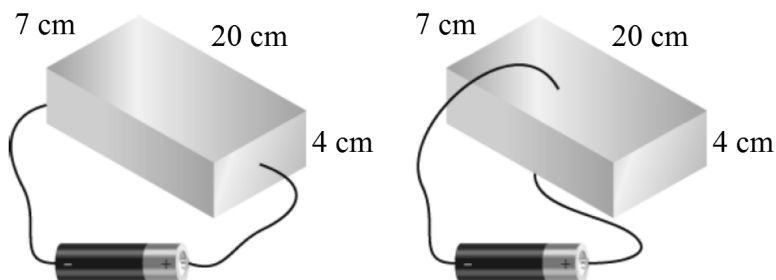
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Using the definition of resistance, and the relationship between resistance and resistivity,

$$I = \frac{\Delta V}{R} \quad \text{and} \quad R = \rho \frac{L}{A}$$

so

$$I = \frac{\Delta V}{\rho(L/A)} = \frac{A \Delta V}{\rho L}$$



Letting the situation with known current in the left diagram be “left”, and the situation with unknown current in the right diagram be “right”,

$$\frac{I_{\text{right}}}{I_{\text{left}}} = \frac{A_{\text{right}} \Delta V_{\text{right}} / (\rho_{\text{right}} L_{\text{right}})}{A_{\text{left}} \Delta V_{\text{left}} / (\rho_{\text{left}} L_{\text{left}})} = \frac{A_{\text{right}} L_{\text{left}}}{A_{\text{left}} L_{\text{right}}}$$

as the potential differences and the resistivities are the same. So

$$I_{\text{right}} = \frac{A_{\text{right}} L_{\text{left}}}{A_{\text{left}} L_{\text{right}}} I_{\text{left}} = \frac{(20 \text{ cm} \times 7 \text{ cm}) (20 \text{ cm})}{(4 \text{ cm} \times 7 \text{ cm}) (4 \text{ cm})} (3.0 \text{ A}) = 75 \text{ A}$$

5. (5 points) The switch S has been closed for a long time. Immediately upon opening the switch, what is the current through the $20\ \Omega$ resistor?

After the switch has been closed for a long time, no current flows through the $60\ \Omega$ resistor. (If there were current flowing through it, the charge on the capacitor would still be increasing, so the switch wouldn't have been closed "for a long time".) Current only flows through the $30\ \Omega$ and $20\ \Omega$ resistors, which have the same current of

$$I_{\text{closed}} = \frac{\Delta V}{R} = \frac{100\ \text{V}}{30\ \Omega + 20\ \Omega} = 2.0\ \text{A}$$

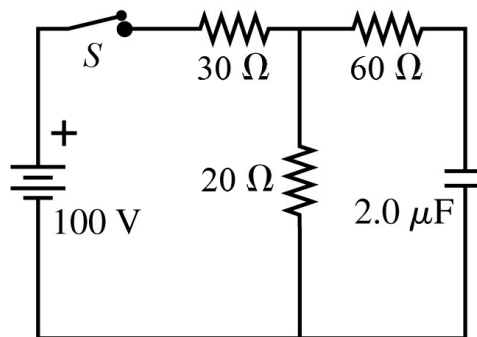
The potential difference across the $20\ \Omega$ resistor is

$$\Delta V = IR = (2.0\ \text{A})(20\ \Omega) = 40\ \text{V}$$

That $40\ \text{V}$ must also be the potential across the capacitor, as there is no potential difference across the $60\ \Omega$ resistor when no current flows through it.

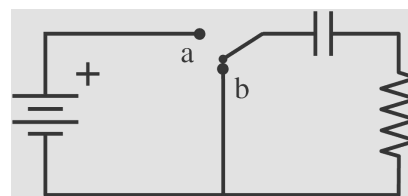
When the switch is opened, no current can flow through the $30\ \Omega$ resistor. Current only flows through the $60\ \Omega$ and $20\ \Omega$ resistors, which have the same current through them. At the instant the switch is opened, the potential across the $60\ \Omega$ and $20\ \Omega$ combination is the $40\ \text{V}$ from the capacitor, so the current through them is

$$I_{\text{opened}} = \frac{\Delta V}{R} = \frac{40\ \text{V}}{60\ \Omega + 20\ \Omega} = 0.50\ \text{A}$$



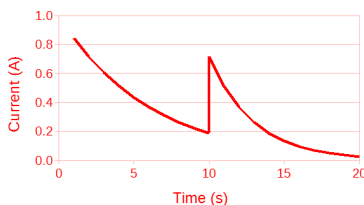
6. (5 points) The capacitor is initially uncharged. The switch is thrown to position a at time $t = 0$, and then to b at time $t = 10\ \text{s}$. Which graph could represent the magnitude of the current through the resistor as a function of time?

When the switch is thrown to position a , the charge on the capacitor begins to increase, so the potential across the capacitor increases. Since the sum of the potentials around the outside loop is zero (Kirchhoff's Loop Law) and the potential across the battery is fixed, the potential across the resistor must decrease. Therefore, the current through the resistor decreases.



When the switch is thrown to position b , the charge on the capacitor begins to decrease, so the potential across the capacitor decreases. Since the sum of the potentials around the right loop is zero (Kirchhoff's Loop Law), the potential across the resistor must decrease as well. Therefore, the current through the resistor decreases.

The only graph showing a current decrease both from $t = 0$ and from $t = 10\ \text{s}$ is



Note the discontinuity introduced by throwing the switch.

7. (5 points) In the circuit shown, the four capacitors are identical: each capacitor has the same capacitance; furthermore, each capacitor has the same rating for the maximum potential that it can withstand without suffering dielectric breakdown (i.e, without “shorting out” across the gap, permanently ruining the capacitor).

Suppose the applied potential difference V_0 is gradually increased from zero until one of the capacitors finally breaks down. Which capacitor is it?

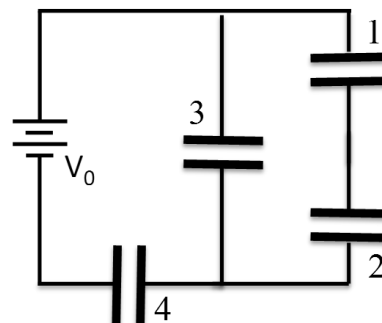
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Capacitor 4 must have a charge equal to the total charge delivered by the battery. Capacitor 4 is in series with the combination of capacitors 1, 2, and 3, but it is not in series with any single one of those capacitors. At every instant, then, capacitor 4 must have more charge than any one of capacitors 1, 2, or 3. From the definition of capacitance,

$$Q = C \Delta V \quad \Rightarrow \quad \Delta V = \frac{Q}{C}$$

capacitor 4 must have a greater potential across it than any one of capacitors 1, 2, or 3. The capacitor that breaks down first is

Capacitor 4



8. (5 points) Two cylindrical conductors of equal length L are connected to a battery, as shown. The conductors are made of the same material, but have different radii. Which conductor, if either, dissipates more power?

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The power dissipated in a resistor can be related to its geometry with the definition of resistance, and the relationship between resistance and resistivity.

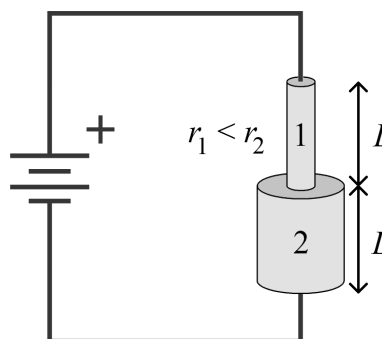
$$P = I \Delta V \quad \text{and} \quad \Delta V = IR \quad \text{so} \quad P = I^2 R$$

and

$$R = \rho \frac{L}{A} \quad \text{so} \quad P = I^2 \rho \frac{L}{A}$$

The two cylindrical conductors have the same current, resistivity, and length, so that with the smallest cross-sectional area will dissipate more power. That would be

Conductor 1.



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9. (5 points) An air-gap parallel-plate capacitor with capacitance C_0 is brought to a charge Q_0 by attaching it to a battery with emf $\mathcal{E}$. With the battery still attached, a slab of dielectric with dielectric constant κ is inserted between the capacitor plates, completely filling the air gap. What are the charge Q on the capacitor, and the potential difference ΔV across it, after the dielectric is inserted?
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The capacitance of a parallel-plate capacitor is

$$C = \epsilon \frac{A}{d} = \kappa \epsilon_0 \frac{A}{d} = \kappa C_0$$

If the battery remains connected to the capacitor, the potential across it cannot change. From the definition of capacitance, the charge changes when a dielectric is inserted, according to

$$Q_0 = C_0 \Delta V_0 = C_0 \mathcal{E} \quad \text{and} \quad Q = C \Delta V = \kappa C_0 \mathcal{E} = \kappa Q_0$$

So

$$Q = \kappa Q_0 \quad \text{and} \quad \Delta V = \mathcal{E}$$

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10. (5 points) An air-gap parallel-plate capacitor with capacitance C_0 is brought to a charge Q_0 by attaching it to a battery with emf $\mathcal{E}$. The battery is then detached. With the battery still detached, a slab of dielectric with dielectric constant κ is inserted between the capacitor plates, completely filling the air gap. What are the charge Q on the capacitor, and the potential difference ΔV across it, after the dielectric is inserted?
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The capacitance of a parallel-plate capacitor is

$$C = \epsilon \frac{A}{d} = \kappa \epsilon_0 \frac{A}{d} = \kappa C_0$$

If the battery is disconnected from the capacitor, the charge on it cannot change. From the definition of capacitance, the potential changes when a dielectric is inserted, according to

$$\Delta V_0 = \frac{Q_0}{C_0} = \mathcal{E} \quad \text{and} \quad \Delta V = \frac{Q}{C} = \frac{Q_0}{\kappa C_0} = \mathcal{E}/\kappa$$

So

$$Q = Q_0 \quad \text{and} \quad \Delta V = \mathcal{E}/\kappa$$