
I. (16 points) An object is subject to acceleration in one dimension of the form

$$a(t) = a_0 \cos(\omega t)$$

where a_0 and ω are positive constants. At time $t = 0$ the object is found at the origin with a velocity of zero. Calculate the next time at which velocity of the object is zero. Express your answer in terms of parameters defined in the problem and physical or mathematical constants.

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This is a kinematics problem, with non-constant acceleration. As the motion is constrained to one dimension, vector notation is unnecessary—signs are sufficient to indicate direction. Find an expression for velocity v from its relationship to acceleration.

$$a = \frac{dv}{dt} \qquad \Rightarrow \qquad v = \int dv = \int a\,dt = \int a_0 \cos(\omega t)\,dt = \frac{1}{\omega}\int a_0 \cos(\omega t)\,\omega\,dt = \frac{a_0}{\omega}\sin(\omega t) + C$$

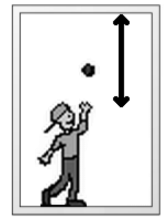
where C is an integration constant. Since $v = 0$ at $t = 0$,

$$v = \frac{a_0}{\omega}\sin(\omega t) + C \qquad \Rightarrow \qquad 0 = \frac{a_0}{\omega}\sin(0) + C \qquad \Rightarrow \qquad C = 0 \qquad \text{so} \qquad v = \frac{a_0}{\omega}\sin(\omega t)$$

As a_0/ω cannot be zero, v can only be zero when $\sin(\omega t) = 0$. This occurs when $\omega t = -\pi$, 0, π, 2π, 3π, etc. So the first time it occurs after $t = 0$ is at

$$\omega t = \pi \qquad \Rightarrow \qquad t = \frac{\pi}{\omega}$$

II. (16 points) Inside a large crate, Billy throws a ball straight upward. It hits the top of the crate with a speed v_f at a distance d above his hand. How much time was required for the ball to travel from Billy's hand to the top of the crate? Express your answer in terms of parameters defined in the problem and physical or mathematical constants. *On Earth.*



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This is free-fall problem, which is a special case of a constant-acceleration kinematics problem in which the constant acceleration is g. Choose a coordinate system. I'll choose the origin at Billy's hand, with positive upward. The two constant-acceleration kinematics expressions worth remembering are:

$$x = x_0 + v_0\,\Delta t + \tfrac{1}{2}a\,(\Delta t)^2$$
$$v = v_0 + a\,\Delta t$$

Eliminate v_0.

$$v_0 = v - a\,\Delta t$$
$$x = x_0 + (v - a\,\Delta t)\,\Delta t + \tfrac{1}{2}a\,(\Delta t)^2 = x_0 + v\,\Delta t - \tfrac{1}{2}a\,(\Delta t)^2$$
$$\tfrac{1}{2}a\,(\Delta t)^2 - v\,\Delta t + (x - x_0) = 0$$

That's a quadratic in Δt, so

$$\Delta t = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

where, with the coordinate system chosen,

$$A = \tfrac{1}{2}a = \tfrac{1}{2}(-g) = -\tfrac{1}{2}g \qquad B = -v = -v_f \qquad C = (x - x_0) = d - 0 = d$$

so

$$\Delta t = \frac{-(-v_f) \pm \sqrt{(-v_f)^2 - 4\left(-\tfrac{1}{2}g\right)d}}{2\left(-\tfrac{1}{2}g\right)} = \frac{-v_f \mp \sqrt{v_f^2 + 2gd}}{g}$$

Select the positive time.

$$\Delta t = \frac{-v_f + \sqrt{v_f^2 + 2gd}}{g}$$

1. (6 points) In the problem above, consider what happens if Billy simply drops the rock, rather than throwing it upward. Compare the acceleration of the rock shortly after leaving Billy's hand when dropped, to the acceleration when he throws it upward.

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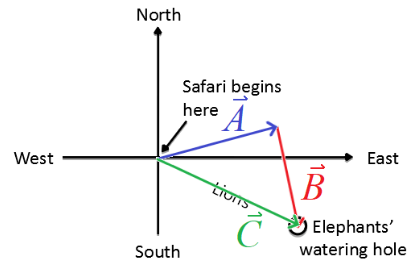
The rock is in free-fall after it leaves Billy's hand, regardless of whether it is thrown upward or simply dropped.

When dropped, the rock has the same acceleration magnitude as when it is thrown upward, in the same direction.

III. (16 points) You are on a safari on the Kalahari, with the ambition of watching elephants. The best place to watch elephants is on their watering hole, 10 kilometers and 30° south of east from your current location. However, to avoid lions between the starting point and the watering hole, you decide to walk 15° north of east for 90 minutes at a speed of 5 kilometers per hour. After taking this detour, what distance and direction must you walk to reach the watering hole? Provide the direction as an angle with respect to the East.



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Let the position of the watering hole be $\vec{C}$. Let your first displacement be $\vec{A}$ and your second displacement be $\vec{B}$. Then $\vec{C} = \vec{A} + \vec{B}$ and $\vec{B}$ is the answer to the question. If x is in the direction of East, and y is in the direction of North, then

$$\vec{B} = \vec{C} - \vec{A} \qquad \Rightarrow \qquad B_x = C_x - A_x \qquad \text{and} \qquad B_y = C_y - A_y$$

So

$$B_x = C_x - A_x = (10\,\text{km})\cos(-30°) - (5\,\text{km/hr})\,(1.5\,\text{hr})\cos(15°) = 1.4\,\text{km}$$

and

$$B_y = C_y - A_y = (10\,\text{km})\sin(-30°) - (5\,\text{km/hr})\,(1.5\,\text{hr})\sin(15°) = -6.9\,\text{km}$$

Therefore

$$B = \sqrt{B_x^2 + B_y^2} = \sqrt{(1.4\,\text{km})^2 + (-6.9\,\text{km})^2} = 7.1\,\text{km}$$

at

$$\tan\theta = \frac{B_y}{B_x} \qquad \Rightarrow \qquad \theta = \tan^{-1}\left(\frac{-6.9\,\text{km}}{1.4\,\text{km}}\right) = -78°$$

so you must walk

7 km 80° south of East

2. (6 points) On the return trip, you walk directly west until you are exactly south of the starting point and then north, until you are back on the safari starting point. What distance did you walk in total on the way back?

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Walking directly west then directly north makes your path the legs of a right triangle. The hypotenuse is the direct line from the watering hole to the safari starting point.

$$C_x = (10\,\text{km})\cos(-30°) = 8.7\,\text{km} \qquad \text{and} \qquad C_y = (10\,\text{km})\sin(-30°) = -5.0\,\text{km}$$

So the total distance walked back is

$$|C_x| + |C_y| = 8.7\,\text{km} + 5.0\,\text{km} = 13.7\,\text{km}$$

3. (8 points) A train slows as it approaches a train station from the right. The station is at the origin, and the positive direction is toward the right, as shown. Unfortunately, the train overshoots the station, and comes to a stop to the left of the station. It then speeds up in reverse, to approach the station from the left. What is the sign of the train's acceleration as it approaches the station, first from the right, and then from the left.

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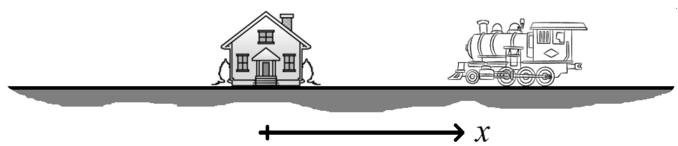
Approaching from the right, the velocity is becoming less negative. Approaching from the left, the velocity is becoming more positive.

Acceleration is positive approaching from the right, and positive approaching from the left.



4. (8 points) Two objects, A and B, are traveling in the positive direction. Object A has constant positive velocity. Object B has positive velocity and a positive acceleration that decays exponentially with time, as shown. Object B passes object A at time t. How does the distance between the objects change with time after that?

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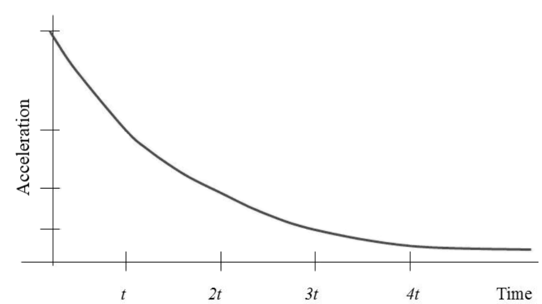
Even though the acceleration of object B is decreasing after it passes object A, the acceleration never reaches zero. That is, object B always has positive acceleration. It's velocity, therefore, is always becoming more and more positive. So

The distance between A and B always increases, so B is always ahead of A by an ever-increasing amount.



5. (8 points) Using the coordinate system shown, which is the correct expression for the force vector $\vec{F}$?

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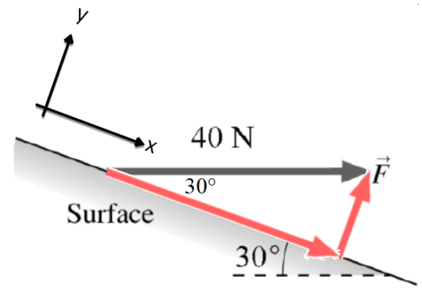

Components of the force vector in that coordinate system are shown.

$$F_x = F\cos(30°) = (40\,\mathrm{N})\cos(30°) = 35\,\mathrm{N}$$

and

$$F_y = F\sin(30°) = (40\,\mathrm{N})\sin(30°) = 20\,\mathrm{N}$$

so

$$\vec{F} = 35\hat{\imath} + 20\hat{\jmath}\,\mathrm{N}$$

6. (8 points) A train and a car are traveling parallel to each other at constant velocity $\vec{v}_0$. At t=0, the car slows down with acceleration magnitude a, while the train continues to travel with constant velocity. As soon as the car reaches speed $v = 0$ it begins to speed up, in the same direction as before, with acceleration magnitude a. By the time that the car reaches velocity $\vec{v}_0$ again, what is the ratio D_t/D_c of the distance covered by the train by the distance covered by the car? *Hint:* This problem is most easily solved graphically.

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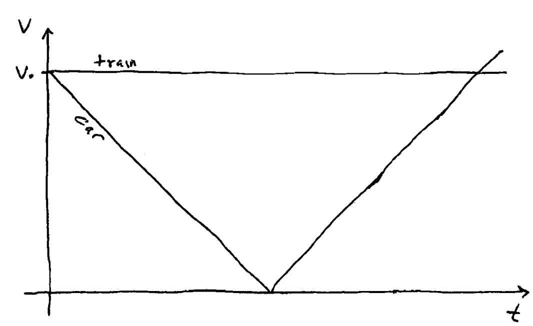
Sketch a graph of the car and train's velocity as a function of time, as shown. The displacement of each is the area under the corresponding curve. The area under the train's curve is a rectangle. The area under the car's curve is two triangles.

$$D_t/D_c = 2$$



7. (8 points) The graph shows the position of an object moving along the x axis as a function of time t. At what time, if any, in the graphed range is the instantaneous velocity equal to the average velocity over the entire range?

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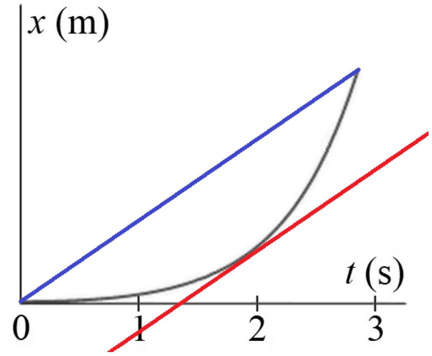

The average velocity is the displacement divided by the time required. Graphically, this is the slope of the line from the initial to the final time on the graph of position vs. time.

The instantaneous velocity is the limit of the average velocity as the time interval approaches zero. Graphically, this is a slope of a line tangent to the graph of position vs. time.

If the average and instantaneous velocities are the same, these two lines must be parallel. A line parallel to the line from the initial to final time is tangent to the curve

near time $t = 2\,\mathrm{s}$.