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g Magnitude of Free Fall Acceleration G Universal Gravitation Constant

Unless otherwise directed, drag should be neglected, and all ropes and pulleys are ideal.
Any integrals in free-response problems must be evaluated. Questions about magnitudes will state so explicitly.

I. (16 points) An object of mass m is to be launched from the surface of a planet with mass M. The planet has radius R and an atmosphere with thickness R_a. This atmosphere exerts a constant drag force D on the object, for as long as the object is in the atmosphere. With what initial speed must the object be launched, if it is never to return to the planet? Express your answer in terms of parameters defined in the problem, and physical or mathematical constants.

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Choose a system consisting of the object, the planet, and it atmosphere, and write the *energy equation* for it.

$$W_{\text{ext}} = \Delta K + \Delta U + \Delta E_{\text{th}}$$

With this choice of system, there are no external forces after the initial launch, so $W_{\text{ext}} = 0$. One may assume the planet is much more massive than the object, so the kinetic energy change of the planet is negligible. The potential energy change of the system is due to the internal conservative force of gravity. The drag force of the atmosphere increases the thermal energy of the system.

$$0 = \left(\tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2\right) + \left(\frac{-GMm}{r_f} - \frac{-GMm}{r_i}\right) + D\,\Delta s$$

If the object is never to return, its final position approaches infinite separation ($r_f \to \infty$), and its final speed can approache zero ($v_f \to 0$). The object starts on the planet's surface, so $r_i = R$. The drag force acts while the object is in the atmosphere, so $\Delta s = R_a$.

$$0 = \left(0 - \tfrac{1}{2}mv_i^2\right) + \left(0 + \frac{GMm}{R}\right) + DR_a$$

Solve for v_i.

$$\tfrac{1}{2}mv_i^2 = \frac{GMm}{R} + DR_a$$

$$v_i^2 = \frac{2GM}{R} + \frac{2DR_a}{m}$$

$$v_i = \sqrt{\frac{2GM}{R} + \frac{2DR_a}{m}}$$

II. (16 points) A non-uniform thin rod lies on the $+x$ axis, with one end at the origin, as shown. It has mass M, length L, cross-sectional area A, and a density ρ that varies with position x according to

$$\rho = \rho_0 \left(\frac{x^2}{L^2} - \frac{x^3}{L^3} \right)$$

where ρ_0 is a positive constant. What is the rod's moment of inertia about the y axis? Express your answer in terms of parameters defined in the problem, and physical or mathematical constants.

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In general, an object's moment of inertia is

$$I = \int r^2 \, dm$$

Consider an element of mass dm that is a slice of the rod, with width dx and area A. The volume of this element is $dV = A\,dx$. The density of the rod is

$$\rho = \frac{dm}{dV} = \frac{dm}{A\,dx} \qquad \Rightarrow \qquad dm = \rho A\,dx$$

Note that there are an infinite number of such elements, each a distance $r = x$ from the y axis. We need to include the effect of all the elements, so the integration limits are $x = 0$ to $x = L$.

$$I = \int r^2 \, dm = \int_0^L x^2 \, \rho A \, dx = \int_0^L x^2 \rho_0 \left(\frac{x^2}{L^2} - \frac{x^3}{L^3} \right) A \, dx = \rho_0 A \int_0^L \left(\frac{x^4}{L^2} - \frac{x^5}{L^3} \right) dx$$

$$= \rho_0 A \left[\int_0^L \frac{x^4}{L^2} \, dx - \int_0^L \frac{x^5}{L^3} \, dx \right] = \rho_0 A \left[\frac{1}{L^2} \int_0^L x^4 \, dx - \frac{1}{L^3} \int_0^L x^5 \, dx \right] = \rho_0 A \left[\frac{1}{L^2} \frac{x^5}{5} \Big|_0^L - \frac{1}{L^3} \frac{x^6}{6} \Big|_0^L \right]$$

$$= \rho_0 A \left[\frac{1}{L^2} \left(\frac{L^5}{5} - \frac{0^5}{5} \right) - \frac{1}{L^3} \left(\frac{L^6}{6} - \frac{0^6}{6} \right) \right] = \rho_0 A \left[\frac{L^3}{5} - \frac{L^3}{6} \right] = \rho_0 A L^3 \left[\frac{6}{30} - \frac{5}{30} \right] = \frac{\rho_0}{30} A L^3$$

1. (6 points) Let the moment of inertia found in the problem above be I. Describe the rod's moment of inertia I' about an axis along the line $x = -d$.

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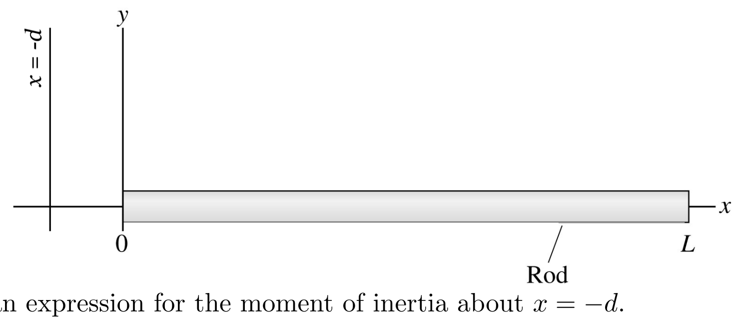

Let the rod's center of mass be at some position $x = x_{\text{CM}}$. Use the parallel axis theorem to find an expression for the moment of inertia about the center of mass.

$$I = I_{\text{CM}} + Mh^2 \qquad \Rightarrow \qquad I_{\text{CM}} = I - Mx_{\text{CM}}^2$$

Use the parallel axis theorem again to find an expression for the moment of inertia about $x = -d$.

$$I' = I_{\text{CM}} + Mh^2 = \left(I - Mx_{\text{CM}}^2 \right) + M \left(x_{\text{CM}} + d \right)^2 = \left(I - Mx_{\text{CM}}^2 \right) + M \left(x_{\text{CM}}^2 + 2x_{\text{CM}} d + d^2 \right) = I + 2Mx_{\text{CM}} d + Md^2$$

$$I' > I + Md^2$$

2. (6 points) A bullet with mass m is shot into a block with mass M, at rest on a frictionless horizontal surface. The bullet remains lodged in the block. The block moves into an ideal spring with Hooke's Law constant K, and compresses it by a distance d. What physical principle can you use to relate the speed of the bullet before it hits the block, to the speed of the bullet-block combination before it hits the spring?

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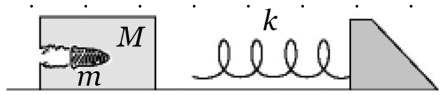

There are no significant net external forces on the bullet-block system during the embedding process. Both energy and momentum are conserved. However, the interaction force between the bullet and the block is non-conservative, and increases the thermal energy of the system. Mechanical energy is not conserved in the system, and there's no reason to expect kinetic energy to be conserved. Although total energy is conserved, insufficient information is provided to determine the change in thermal energy, so conservation of energy, despite being valid, is not a useful approach.

Nothing suggests that this interaction force is constant, which would be necessary for constant acceleration kinematics to be used. The appropriate principle, then, is

Conservation of linear momentum in the bullet-block system.

III. (16 points) In the problem above, what was the speed of the bullet before it hit the block? Express your answer in terms of parameters defined in the problem, and physical or mathematical constants.

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First, find the speed of the bullet-block combination after the bullet embeds itself in the block. Momentum is conserved in the bullet-block system. Let the bullet be traveling in the positive direction.

$$\vec{p}_f = \vec{p}_i \qquad \Rightarrow \qquad mv_i + MV_i = mv_f + MV_f \qquad \Rightarrow \qquad mv_i + 0 = (m+M)\,V_f \qquad \Rightarrow \qquad V_f = \frac{m}{m+M} v_i$$

where i and f denote initial and final states. Now the *energy equation* can be written for the bullet-block-spring system.

$$W_{\text{ext}} = \Delta K + \Delta U + \Delta E_{\text{th}}$$

There are no external forces doing work on this system, and once the embedding process is over, there are no internal dissipative forces changing the thermal energy. The internal elastic force of the spring changes the system's potential energy, and the kinetic energy of the bullet-block combination changes.

$$0 = \left(\tfrac{1}{2}m'v_f'^2 - \tfrac{1}{2}m'v_i'^2\right) + \left(\tfrac{1}{2}k\,(\Delta s_f)^2 - \tfrac{1}{2}k\,(\Delta s_i)^2\right) + 0$$

where m' is the mass of the bullet-block combination $m + M$, v' is the speed of the bullet-block combination (note that the speed at the beginning of the sliding process is the same as the speed at the end of the embedding process, $v_i' = V_f$).

$$0 = \left(0 - \tfrac{1}{2}\,(m+M)\,V_f^2\right) + \left(\tfrac{1}{2}kd^2 - 0\right) \qquad \Rightarrow \qquad \tfrac{1}{2}\,(m+M)\,V_f^2 = \tfrac{1}{2}kd^2 \qquad \Rightarrow \qquad (m+M)\left(\frac{m}{m+M}v_i\right)^2 = kd^2$$

Solve for v_i.

$$\frac{m^2 v_i^2}{m+M} = kd^2 \qquad \Rightarrow \qquad v_i^2 = \frac{m+M}{m^2}kd^2 \qquad \Rightarrow \qquad v_i = \frac{d}{m}\sqrt{k\,(m+M)}$$

3. (6 points) Which of the graphs represents a spring that gets less stiff the more it is stretched?

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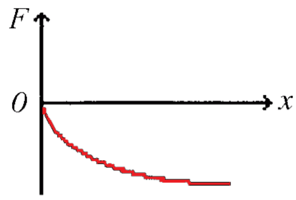
A spring's Hooke's Law constant, k, is a measure of its stiffness. For an ideal spring, k is constant and is the magnitude of the slope of the graph of force as a function of position. For a real spring, k is still the magnitude of the slope, but is not necessarily constant. This question is asking about a spring whose graph of force as a function of position has decreasing slope magnitude as the spring is stretched.



4. (7 points) Swimmers at a water park have a choice of two frictionless water slides as shown in the figure. Although both slides drop over the same height h, Slide 1 is straight while Slide 2 is curved, dropping quickly at first and then leveling out. If it can be determined, how does the speed v_1 of a swimmer reaching the end of Slide 1 compare with v_2, the speed of a swimmer reaching the end of Slide 2? *(On Earth.)*

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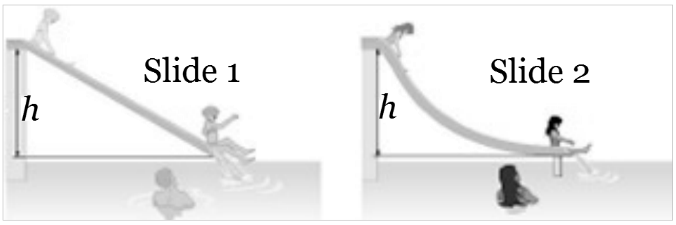
With no external forces acting on, and no dissipative forces acting in, each Earth-swimmer system, mechanical energy is conserved.

$$0 = \Delta K + \Delta U \qquad \Rightarrow \qquad 0 = \left(\tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2\right) + mg\,\Delta h$$



Swimmers start at rest, and have the same height change.

$$0 = \left(\tfrac{1}{2}mv_f^2 - 0\right) + mg\,\Delta h \qquad \Rightarrow \qquad v_f^2 = -2g\,\Delta h \qquad \Rightarrow \qquad v_1 = v_2$$

5. (7 points) A dumbbell-shaped object is composed by two equal masses m, connected by a rod of negligible mass and length r. If I_1 is the moment of inertia of this object with respect to an axis passing through the center of the rod and perpendicular to it, and I_2 is the moment of inertia with respect to an axis passing through one of the masses . . .

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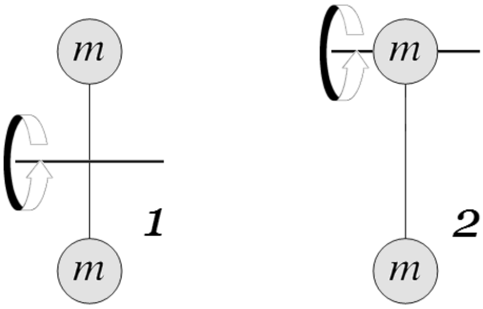
I_1 is through the center of mass, so the parallel axis theorem can be applied.

$$I = I_{\text{cm}} + Md^2 \qquad \Rightarrow \qquad I_2 = I_1 + M\left(\frac{r}{2}\right)^2 \qquad \Rightarrow \qquad I_1 < I_2$$



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6. (7 points) A uniform solid sphere of mass M and radius R rotates with an angular speed ω about an axis through its center. A uniform solid cylinder of mass M, radius R, and length $2R$ rotates through an axis running through the central axis of the cylinder. What must be the angular speed of the cylinder so it will have the same rotational kinetic energy as the sphere?

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The rotational kinetic energy of an object is $\frac{1}{2}I\omega^2$. Letting "c" represent the cylinder and "s" represent the sphere,

$$K_{\text{c}} = K_{\text{s}} \qquad \Rightarrow \qquad \tfrac{1}{2}I_{\text{c}}\omega_{\text{c}}^2 = \tfrac{1}{2}I_{\text{s}}\omega_{\text{s}}^2 \qquad \Rightarrow \qquad \omega_{\text{c}} = \sqrt{\frac{I_{\text{s}}}{I_{\text{c}}}}\,\omega_{\text{s}}$$

From the table

$$I_{\text{s}} = \frac{2}{5}MR^2 \qquad \text{and} \qquad I_{\text{c}} = \frac{1}{2}MR^2$$

Note that M and R are the same for the sphere and cylinder in this question, and ω_{s} is just ω. So

$$\omega_{\text{c}} = \sqrt{\frac{\frac{2}{5}MR^2}{\frac{1}{2}MR^2}}\,\omega = \frac{2\omega}{\sqrt{5}}$$

7. (7 points) Jacques and George have met in their canoes at the middle of a lake. They stopped, and when they are ready to leave, Jacques pushes George's canoe with a force $\vec{F}$ to separate the two canoes. As the canoes drift apart, what is true about the momentum and kinetic energy of the system consisting of Jacques, George, and the two canoes, if we can neglect any resistance due to the water? *(On Earth.)*

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The momentum of the system is zero before Jacques pushes George's canoe. Since Jacques' push force is internal to the system, it cannot change the momentum of the system. Being internal, it cannot change the *total* energy of the system, either, but the form of energy can change. In this case, chemical energy in Jacques' body is transformed into kinetic energy (the canoes move).

Note that momentum is a vector, so the canoes moving in opposite directions can result in a momentum of zero. However, kinetic energy is a positive scalar, so two non-zero kinetic energies cannot sum to zero.

The momentum is zero, but the kinetic energy is positive.

8. (6 points) An athlete holds a ball in her hand. *(On Earth.)* The torque she must apply about her shoulder joint to hold the ball straight out to her side is ...

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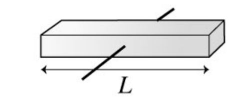
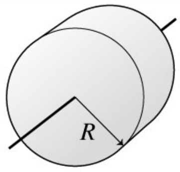
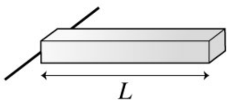
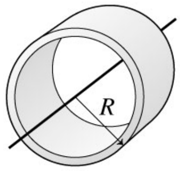
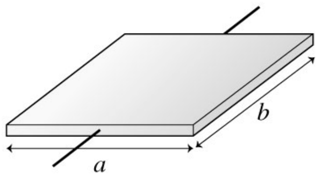
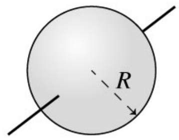
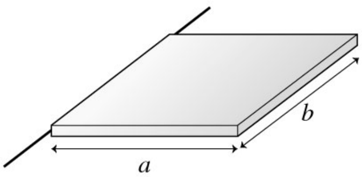
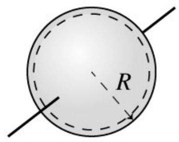
If the ball is to be held in place, the net torque about the shoulder joint must be zero. The torque applied by the athlete must balance the gravitational torque.

$$\vec{\tau} = \vec{r} \times \vec{F} \qquad \Rightarrow \qquad \tau = rF\sin\phi = rmg\sin\phi$$

where r, m, and g will be the same in both situations. The angle between $\vec{r}$ and $\vec{F}$ is $90°$ when the ball is held straight out, and $45°$ when the ball is held at $45°$ below the horizontal. $\sin 90° > \sin 45°$ so the torque she must apply about her shoulder joint to hold the ball straight out to her side is

greater than the torque she must apply to hold the ball $45°$ below the horizontal.

TABLE 12.2 Moments of inertia of objects with uniform density

Object and axis	Picture	I	Object and axis	Picture	I
Thin rod, about center		$\frac{1}{12}ML^2$	Cylinder or disk, about center		$\frac{1}{2}MR^2$
Thin rod, about end		$\frac{1}{3}ML^2$	Cylindrical hoop, about center		MR^2
Plane or slab, about center		$\frac{1}{12}Ma^2$	Solid sphere, about diameter		$\frac{2}{5}MR^2$
Plane or slab, about edge		$\frac{1}{3}Ma^2$	Spherical shell, about diameter		$\frac{2}{3}MR^2$