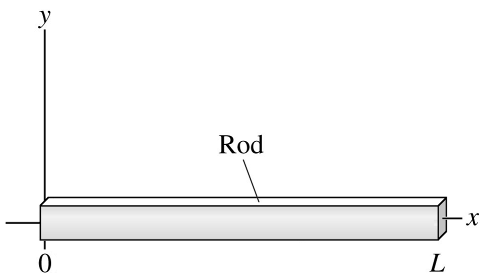
g Magnitude of Free Fall Acceleration $\qquad$ G Gravitational Constant

Unless otherwise directed, drag should be neglected, and all ropes and pulleys are ideal.
Any integrals in free-response problems must be evaluated. Questions about magnitudes will state so explicitly.

I. (16 points) A thin rod of mass M and length L lies on the x axis, as shown. Its linear mass density (mass per unit length), λ, depends on position, x, according to

$$\lambda = \lambda_0 \left(\frac{L}{x}\right)$$

where λ_0 is a constant. Find the location of the rod's center of mass, in terms of parameters defined in the problem, and physical or mathematical constants.



The center of mass of a continuous object is located at

$$\vec{r}_{\text{CM}} = \frac{\int \vec{r}\, dm}{\int dm}$$

The denominator is the total mass of the rod.

$$\int dm = M$$

Since the rod is thin and lies on the x axis, $\vec{r} = x$, where the sign of x would indicate the direction. The linear mass density is the mass per unit length

$$\lambda = \frac{dm}{dx} \qquad \Rightarrow \qquad dm = \lambda\, dx = \lambda_0 \left(\frac{L}{x}\right) dx$$

So

$$\int \vec{r}\, dm = \int_0^L x\, \lambda_0 \left(\frac{L}{x}\right) dx = \lambda_0 L \int_0^L dx = \lambda_0 L x\Big|_0^L = \lambda_0 L\,(L-0) = \lambda_0 L^2$$

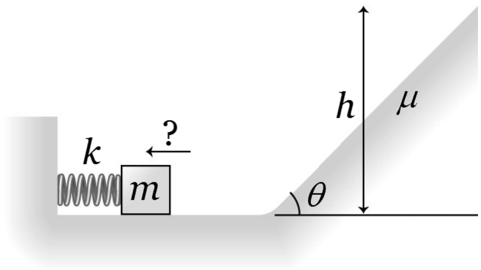
putting the center of mass at

$$x_{\text{CM}} = \lambda_0 L^2 / M$$

which you will note is dimensionally correct.

Students who made a case for the center of mass being at $x = 0$, due to the unrealistic linear mass density approaching infinity as the position approached zero, earned full credit.

II. (16 points) The spring in the figure has a spring constant of k. It is compressed and then launches a block of mass m. The horizontal surface is frictionless, but the block's coefficient of kinetic friction with the incline is μ. This incline rises a height h at an angle θ above the horizontal. What distance must the spring be compressed if the block is to come to a stop at the very top of the incline? Express your answer in terms of parameters defined in the problem, and physical or mathematical constants. *On Earth.*



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Use the Work-Energy Theorem:

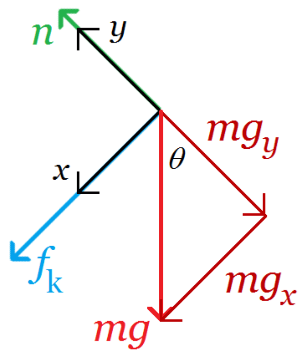
$$W_{\text{ext}} + W_{\text{nc}} = \Delta K + \Delta U$$

Choose a system. I'll include the Earth, the spring, the block, and the surface to be a system. With this choice, there are no external forces, so $W_{\text{ext}} = 0$. Friction is the non-conservative force dissipating mechanical energy in the system, so $W_{\text{nc}} = \vec{f}_{\text{k}} \cdot \vec{x}$. The block is stationary at both its release point and at its highest point, so $\Delta K = 0$. Both elastic and gravitational potential energies need to be considered.

$$0 + -f_{\text{k}} x = 0 + \left[\tfrac{1}{2}k\left(\Delta s_f\right)^2 - \tfrac{1}{2}k\left(\Delta s_i\right)^2\right] + \left[mgh_f - mgh_i\right]$$

where the block slides a distance $x = h/\sin\theta$ along the incline. I'll choose the initial height to be zero, making $h_f = h$. The spring isn't compressed after the block is launched.

$$-\mu n \frac{h}{\sin\theta} = \left[0 - \tfrac{1}{2}k\left(\Delta s_i\right)^2\right] + \left[mgh - 0\right]$$



Use Newton's Second Law to find the normal force n from the incline on the block. In the y direction

$$\sum F_y = n - mg_y = ma_y = 0 \qquad \Rightarrow \qquad n = mg_y = mg\cos\theta$$

Substituting

$$-\mu\left(mg\cos\theta\right)\frac{h}{\sin\theta} = -\tfrac{1}{2}k\left(\Delta s_i\right)^2 + mgh$$

Solve for the initial compression Δs_i.

$$\tfrac{1}{2}k\left(\Delta s_i\right)^2 = mgh + \mu mg\frac{h}{\tan\theta} \qquad \Rightarrow \qquad \left(\Delta s_i\right)^2 = \frac{2mgh}{k}\left(1 + \frac{\mu}{\tan\theta}\right)$$

$$\Delta s_i = \sqrt{\frac{2mgh}{k}\left(1 + \frac{\mu}{\tan\theta}\right)}$$

1. (5 points) The block stops at a height h above the horizontal surface in the problem above. If the angle of the incline were reduced to $\theta/2$, the block stops at a height h' above that horizontal surface. Compare the heights h and h'.

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Consider the extreme angles, $\theta = 0°$ and $\theta = 90°$. When $\theta = 90°$, there is no normal force, so there is no frictional force. All of the energy stored in the spring is transformed to gravitational potential energy. h is large. When $\theta = 0°$, the block slides to a stop along a level surface. h is zero. Therefore, as the angle θ gets smaller, h gets smaller as well.

$$h' < h$$

2. (5 points) A thin ring with mass M and radius R is pivoted about an axle through it's edge, and perpendicular to its face. Exactly opposite the pivot point, a thin rod of mass M and length R is attached, as shown. What is the moment of inertia of this combined object about the axle?

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The ring and rod rotate about the same axis, so the moment of inertia of the combined object is the sum of the moments of inertia of the ring and rod separately.

From the table, the moment of inertia of a cylindrical hoop about its center axis is

$$I_{\text{ring,CM}} = MR^2$$

Note that a hoop is the same as a ring, as the distribution of mass parallel to the axis does not matter. This ring, of course, is not rotating about its center of mass, so the parallel axis theorem must be used.

$$I_{\text{ring}} = I_{\text{ring,CM}} + Md^2 = MR^2 + MR^2 = 2MR^2$$

where the axis offset d is the one radius R.

Again from the table, the moment of inertia of a thin rod about its center axis is

$$I_{\text{rod,CM}} = \frac{1}{12}ML^2$$

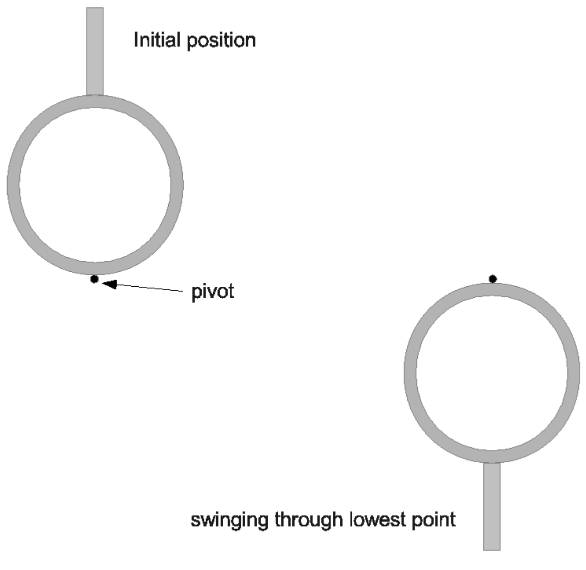
The length of this rod is R, so $L = R$. This rod is not rotating about its center of mass, either, so the parallel axis theorem must be used.

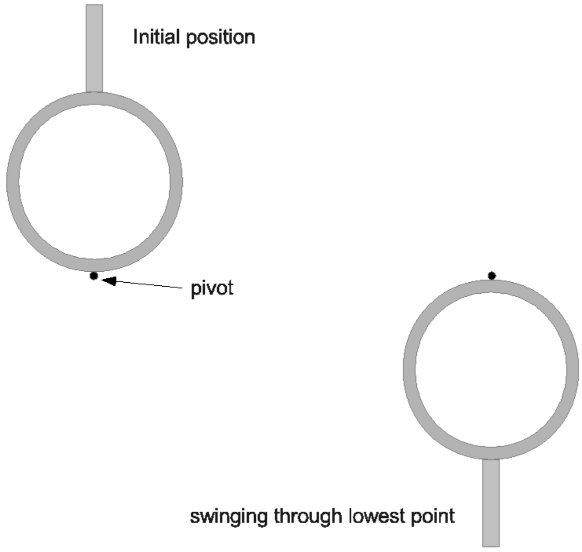
$$I_{\text{rod}} = I_{\text{rod,CM}} + Md^2 = \frac{1}{12}MR^2 + M\left(\frac{5}{2}R\right)^2 = \left(\frac{1}{12} + \frac{25}{4}\right)MR^2 = \frac{76}{12}MR^2$$

where the axis offset d is the half the length of the rod, plus the diameter of the ring.

Adding the two moments of inertia

$$I_{\text{total}} = I_{\text{ring}} + I_{\text{rod}} = 2MR^2 + \frac{76}{12}MR^2 = \frac{24}{12}MR^2 + \frac{76}{12}MR^2 = \frac{100}{12}MR^2 = \frac{25}{3}MR^2$$





III. (16 points) In the problem above, the axle is parallel to the ground. The object is initially at rest with the rod directly above the axle as shown. A small disturbance caused the object to rotate clockwise, as shown. What is the speed of the end of the rod farthest from the axle, at the moment it is directly below the axle? Express your answers in terms of parameters defined in the problem, and physical or mathematical constants. You may use "I" to represent the moment of inertia found in the problem above. *On Earth.*

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Use the Work-Energy Theorem:

$$W_{\text{ext}} + W_{\text{nc}} = \Delta K + \Delta U$$

Choose a system. I'll include the Earth and the object to be a system. With this choice, there are no external forces, so $W_{\text{ext}} = 0$. Nor are there any non-conservative forces dissipating mechanical energy within the system, so $W_{\text{nc}} = 0$. The conservative force of gravity is acting within the system, so gravitational potential energy must be considered. The kinetic energy change of the Earth is negligible, so the only relevant kinetic energy change is that of the object.

$$0 + 0 = \left(\tfrac{1}{2} I\omega_f^2 - \tfrac{1}{2} I\omega_i^2\right) + (mgh_f - mgh_i)$$

The initial kinetic energy of the object is zero. I'll choose the zero of height to be at the pivot point. Remember that the position whose height change must be considered, is the center of mass of the object. Since the ring and the rod are simple shapes, each has its center of mass in its own geometric center. The center of mass can be calculated, then, as if they were two point masses. Putting the origin at the pivot

$$h_i = \frac{m_{\text{ring}} h_{\text{ring,CM}} + m_{\text{rod}} h_{\text{rod,CM}}}{m_{\text{ring}} + m_{\text{rod}}} = \frac{MR + M\,(5/2)\,R}{M + M} = \frac{(7/2)\,R}{2} = \frac{7}{4}R$$

Similarly, $h_f = -\frac{7}{4}R$. So, remembering that the total mass of the object is $2M$,

$$0 = \left(\tfrac{1}{2} I\omega_f^2 - 0\right) + 2Mg\left(\frac{-7}{4}R - \frac{7}{4}R\right) \qquad \Rightarrow \qquad \tfrac{1}{2} I\omega_f^2 = 7MgR \qquad \Rightarrow \qquad \omega_f = \sqrt{\frac{14MgR}{I}}$$

The end of the rod is moving in a circle of radius $3R$. Relating rotational and translational speeds,

$$v = r\omega = (3R)\sqrt{\frac{14MgR}{I}}$$

3. (7 points) Four point masses, $m_A = 1\,\text{kg}$, $m_B = 2\,\text{kg}$, $m_C = 3\,\text{kg}$, and $m_D = 2\,\text{kg}$, are located at the vertexes of a massless square with $2\,\text{m}$ edges, as shown. What is the moment of inertia of this object about the x axis?

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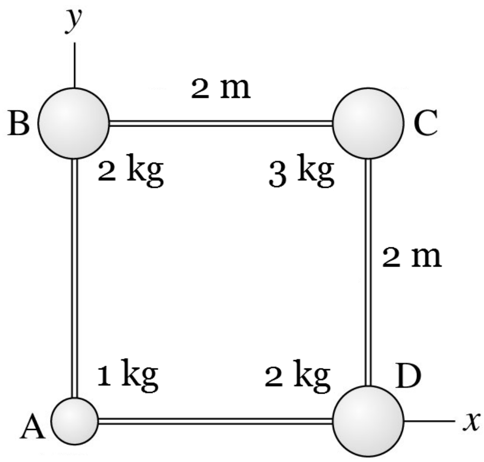

The moment of inertia for a system of point masses is

$$I = \sum m_i r_i^2$$

where r_i is the distance of each mass from the rotation axis. In this case

$$\begin{aligned} I &= m_\text{A} r_\text{A}^2 + m_\text{B} r_\text{B}^2 + m_\text{C} r_\text{C}^2 + m_\text{D} r_\text{D}^2 \\ &= (1\,\text{kg})\,(0\,\text{m})^2 + (2\,\text{kg})\,(2\,\text{m})^2 + (3\,\text{kg})\,(2\,\text{m})^2 + (2\,\text{kg})\,(0\,\text{m})^2 \\ &= 20\,\text{kg·m}^2 \end{aligned}$$

4. (7 points) Uniform solid cylinders A and B have mass $2M$ and radius R. Uniform solid cylinders C and D have mass M and radius $2R$. Cylinders A and C rotate on a axis perpendicular to the page that passes through their centers. Cylinders B and D rotate on a axis perpendicular to the page that passes through their edges. Rank the moments of inertia of the cylinders about their axes, from greatest to least.

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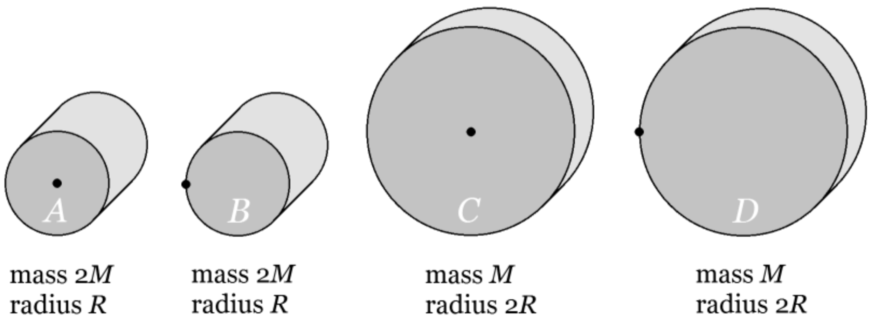
From the table, the moment of inertia of a uniform solid cylinder or disk about its symmetry axis is $I_\text{CM} = \frac{1}{2}mr^2$.

Cylinder A does rotate about its axis through the center of mass, with mass $2M$ and radius R, so



$$I_A = \tfrac{1}{2}\,(2M)\,R^2 = MR^2$$

Cylinder B also has mass $2M$ and radius R, but does not rotate about an axis through its center of mass. The parallel axis theorem must be used.

$$I_B = \tfrac{1}{2}\,(2M)\,R^2 + (2M)\,R^2 = 3MR^2$$

Cylinder C rotates about its axis through the center of mass, but with mass M and radius $2R$, so

$$I_C = \tfrac{1}{2}M\,(2R)^2 = 2MR^2$$

Cylinder D also has mass M and radius $2R$, but does not rotate about an axis through its center of mass. Again, the parallel axis theorem must be used.

$$I_D = \tfrac{1}{2}M\,(2R)^2 + M\,(2R)^2 = 6MR^2$$

Therefore

$$I_D > I_B > I_C > I_A$$

5. (7 points) George P. Burdell has attempted to be the first human being to perform a gravitational slingshot around Mars. The initial speed of the spaceship as seen from a stationary observer is v. The same observer sees Mars moving in the opposite direction with speed U. After the collision, the spaceship and Mars are moving in the same direction, and we can assume the collision to be perfectly elastic. What is the speed of Burdell's spaceship after the collision?

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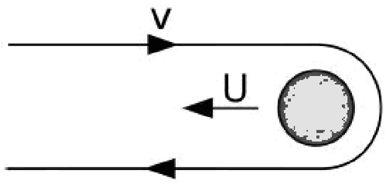

An observer on Mars sees Burdell approaching at a speed $U + v$. Since Mars is much more massive than a spaceship, the speed of Mars is affected negligibly by the elastic collision (remember that a "collision" requires only a period of intense interaction, not physical contact). Since the collision is elastic, then, after the collision the observer on Mars will see Burdell receding at the same speed $U + v$. The stationary observer must see Burdell moving at a speed $U + v$ faster than Mars, or

$$2U + v$$

6. (7 points) When solidly built German Porsche is traveling at a speed of 360 km/h, its engine provides an accelerating force of 4 kN. How much power does the engine deliver in this situation?

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Power is the time rate of energy transformation, $P = dE/dt$. In this case the energy transformation is the work $W = Fs$ done by the engine. Remembering that $1\,\mathrm{W} = 1\,\mathrm{J/s} = 1\,\mathrm{N{\cdot}m/s}$,

$$P = \frac{d\,(Fs)}{dt} = F\frac{ds}{dt} = Fv = (4\,\mathrm{kN})\left(360\frac{\mathrm{km}}{\mathrm{h}} \times \frac{1\,\mathrm{h}}{3.6\,\mathrm{ks}}\right) = 400\,\mathrm{kW}$$

7. (7 points) The potential energy of a system depends on the location of an object within it, according to the graph. What is the force acting on an object when it is at $x = 11\,\text{m}$?

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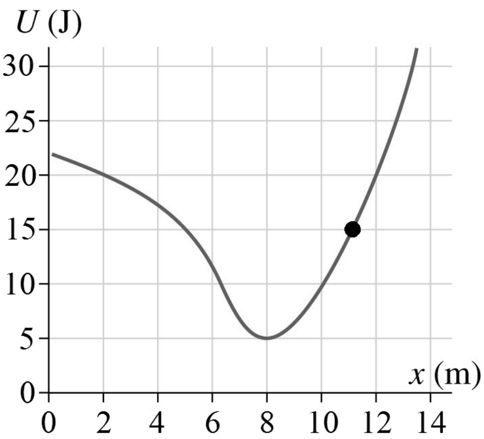
Force and potential energy are related by

$$F_s = -\frac{dU}{ds}$$

so the force is the opposite of the slope of the graph at $x = 11\,\text{m}$.

The graph is approximately linear from $x = 10\,\text{m}$ to $x = 12\,\text{m}$, and rises from $U = 10\,\text{J}$ to $U = 20\,\text{J}$. The force can be approximated by

$$F_s \approx -\frac{\Delta U}{\Delta s} = -\frac{20\,\text{J} - 10\,\text{J}}{12\,\text{m} - 10\,\text{m}} = -5\,\text{N}$$



8. (7 points) The potential energy of a system depends on the location of an object within it, according to the graph. At which location, i–v, does the maximum positive force act on the object?

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Force and potential energy are related by

$$F_s = -\frac{dU}{ds}$$

so the maximum positive force will occur where the graph has maximum negative slope.

Position v.