Fundamental Charge $e = 1.602 \times 10^{-19} \text{C}$
Mass of an Electron $m_e = 9.109 \times 10^{-31} \text{kg}$
Coulomb constant $K = 8.988 \times 10^9 \text{N} \cdot \text{m}^2/\text{C}^2$
Vacuum Permittivity $\epsilon_0 = 8.854 \times 10^{-12} \text{C}^2/\text{N} \cdot \text{m}^2$
Earth’s gravitational field $g = 9.81 \text{N/kg}$
Vacuum Permeability $\mu_0 = 4\pi \times 10^{-7} \text{T} \cdot \text{m/A}$

Unless otherwise directed, friction and drag should be neglected.

Any integrals in free-response problems must be evaluated. Questions about magnitudes will state so explicitly.

I. (18 points) The circuit shown consists of a battery of emf $\mathcal{E} = 24 \text{ V}$ and internal resistance $r = 1.0 \Omega$, and five identical resistors of resistance $R = 8.0 \Omega$. Calculate the current through the resistor $R$ indicated with a star.

Find the equivalent resistance, so the total current supplied by the battery may be determined. The resistors $R$ have been numbered, as shown, for reference.

The potentials across resistors $R_2$ and $R_3$ are the same, so $R_2$ and $R_3$ are in parallel. The combination has resistance
\[ R_{23} = \left( \frac{1}{R_2} + \frac{1}{R_3} \right)^{-1} = \left( \frac{1}{R} + \frac{1}{R} \right)^{-1} = R/2 \]

Similarly, the potentials across resistors $R_4$ and $R_5$ are the same, so $R_4$ and $R_5$ are in parallel. That combination has resistance
\[ R_{45} = \left( \frac{1}{R_4} + \frac{1}{R_5} \right)^{-1} = \left( \frac{1}{R} + \frac{1}{R} \right)^{-1} = R/2 \]

The resistors $r$, $R_1$, $R_{23}$, and $R_{45}$ all have the same current, and so are in series. The equivalent resistance is
\[ R_{eq} = r + R_1 + R_{23} + R_{45} = r + R + \frac{R}{2} + \frac{R}{2} = r + 2R = 1.0 \Omega + 2(8.0 \Omega) = 17.0 \Omega \]

The current supplied by the battery, then, is
\[ \Delta V = I_{eq} R_{eq} \quad \Rightarrow \quad I_{eq} = \frac{\mathcal{E}}{R_{eq}} = \frac{24 \text{ V}}{17.0 \Omega} = 1.41 \text{ A} \]

That current all passes through $R_1$, but must split between $R_2$ and $R_3$. From symmetry considerations, it must divide evenly, and it must continue to be divided evenly as it passes through $R_4$ and $R_5$. The current through the indicated resistor $R_4$, then, is
\[ I^* = I_4 = \frac{I_{eq}}{2} = \frac{1.41 \text{ A}}{2} = 0.71 \text{ A} \]
II. (16 points) A thin horizontal rod with length 30.0 cm is hanging in equilibrium from two identical ideal springs, 10.0 cm apart, as shown. It has a mass of 75.0 mg and is centered in a 2.50 T magnetic field directed out of the page. A current will be passed down one spring, through the rod, and up the other spring. What current (magnitude and direction through the rod) is required to double the stretch of the springs? (On Earth.)

Apply Newton’s Second Law to the rod. Sketch a Free Body Diagram. Before current flows, there is a total spring force magnitude $F_s$ upward, and a gravitational force $mg$ downward. I’ll choose upward to be the positive $x$ direction, and write out Newton’s Second Law in terms of magnitudes.

$$\sum F_x = F_s - mg = ma_x = 0 \quad \Rightarrow \quad F_s = mg$$

When current is flowing, the upward force doubles to $2F_s$ (remember Hooke’s Law, that the force exerted by a spring is proportional to its stretch or compression), and there are a gravitational force $mg$ and a magnetic force $F_B$ downward.

$$\sum F_x = 2F_s - mg - F_B = ma_x = 0 \quad \Rightarrow \quad F_B = 2F_s - mg = 2 (mg) - mg = mg$$

The force on a current-carrying wire in a magnetic field is

$$\vec{F}_B = I\vec{\ell} \times \vec{B} \quad \Rightarrow \quad F_B = I\ell B \sin \phi$$

As the wire lies in the plane of the page and the magnetic field is directed out of the page, the angle $\phi$ is $90^\circ$. The magnetic force is only due to the current is the wire, so the length $\ell$ is 10 cm, not the length of the wire or the width of the field.

$$I = \frac{F_B}{\ell B \sin \phi} = \frac{mg}{\ell B} = \frac{(75.0 \text{ mg} \times 10^{-6} \text{ kg/mg}) (9.81 \text{ N/kg})}{(0.100 \text{ m}) (2.50 \text{ T})} = 0.00294 \text{ A} = 2.94 \text{ mA}$$

Use the Right-Hand-Rule to determine the direction. If the magnetic force (thumb) is to be downward, and the fingers are to curl in the direction of the magnetic field (out of the page), the fingers must have started by pointing

$$\text{to the right.}$$

1. (5 points) If the current found in the problem above is reversed in direction, once the rod stops moving, it’s new position will be . . .

Originally, the spring force upward must balance the gravitational force downward. To double the stretch when current flows, the additional force downward (the magnetic force) must be equal to the original force downward (the gravitational force).

When the current is reversed, the direction of the magnetic force is reversed. With a magnetic force upward that is equal in magnitude to the gravitational force downward, no spring force is required for the rod to be in equilibrium. The rod’s new position will be

$$\text{where the springs are at their relaxed length.}$$
III. (16 points) A long, straight conducting wire of radius \( R \) has a nonuniform current density magnitude

\[ J(r) = J_0 \left(1 - \frac{r^2}{R^2}\right) \]

where \( r \) is the distance from the center of the wire and \( J_0 \) is a positive constant. Find an expression for the magnitude of the magnetic field inside the wire, at radius \( r < R \), in terms of parameters defined in the problem and physical or mathematical constants.

Use Ampere’s Law. Since we wish to find the magnetic field at a distance \( r \) from the center of the wire, choose an Amperian Loop that is a circle with radius \( r \). That loop follows a field line, so the magnetic field \( \vec{B} \) has constant magnitude, and is parallel to the path of the loop at every point.

\[ \oint \vec{B} \cdot d\vec{s} = \mu_0 I_{\text{thru}} \]

\[ \Rightarrow \oint B \cos \theta \, ds = B \cos 0 \oint ds = B 2\pi r = \mu_0 I_{\text{thru}} \]

The current through the loop can be determined by the relationship between current and current density.

\[ I_{\text{thru}} = \oint \vec{J} \cdot d\vec{A} = \int J \cos \phi \, dA \]

Since the current density varies with distance \( r \) from the center of the wire, the area elements \( d\vec{A} \) should be chosen to be “small” in that direction. Thin rings of circumference \( 2\pi r \) and width \( dr \) are appropriate. Choose the direction so \( d\vec{A} \) is parallel to \( \vec{J} \). Choose the integration limits so that all current passing though the Amperian Loop is included.

\[ I_{\text{thru}} = \int J \cos 0 \, dA = \int_0^r J_o \left(1 - \frac{r^2}{R^2}\right) 2\pi r \, dr = 2\pi J_o \int_0^r \left(1 - \frac{r^2}{R^2}\right) r \, dr \]

\[ = 2\pi J_o \left[ \int_0^r r \, dr - \int_0^{r^3} \frac{R^2}{2} \, dr \right] = 2\pi J_o \left( \frac{r^2}{2} \Big|_0^r - \frac{r^4}{4R^2} \Big|_0^r \right) = 2\pi J_o \left( \frac{r^2}{2} - \frac{r^4}{4R^2} \right) \]

So

\[ B 2\pi r = \mu_0 I_{\text{thru}} = B 2\pi r = \mu_0 2\pi J_o \left( \frac{r^2}{2} - \frac{r^4}{4R^2} \right) \Rightarrow B = \mu_0 J_o \left( \frac{r}{2} - \frac{r^3}{4R^2} \right) \]

2. (5 points) What is the expression for the magnitude of the magnetic field outside the wire, at radius \( r > R \)?

From Ampere’s Law

\[ \oint \vec{B} \cdot d\vec{s} = \mu_0 I_{\text{thru}} \]

the magnetic field must be \( \mu_0 \) times something with dimensions of current per length (or, equivalently, current density times length). The only choice offered that has these dimensions is

\[ \mu_0 J_o R^2 / (4r) \]
3. (5 points) Three different ordinary household lightbulbs, A, B, and C, are connected to a battery of emf $E = 40 \text{ V}$ as shown. They are labelled with a nominal power $P_A = 10 \text{ W}$, $P_B = 100 \text{ W}$ and $P_C = 50 \text{ W}$. Rank the brightness of each lightbulb in decreasing order, from the brightest to the faintest. Note: “Nominal” power is what is written on the bulb.

The power dissipated in a resistor is

\[ P = I \Delta V = I^2 R = \frac{(\Delta V)^2}{R} \]

The nominal power rating of a lightbulb is the power dissipated when a normal household potential is placed across it. That would be the same for all the bulbs, so $P = (\Delta V)^2 / R$ the higher the nominal power of a bulb, the less its resistance must be.

Bulbs A and B are in series, and so have the same current. Bulb A has a lower nominal power than bulb B, and therefore a higher resistance, so $(P = I^2 R)$ bulb A dissipates more power, and is brighter than, bulb B.

Bulb C has a higher nominal power than bulb A, and so has lower resistance than bulb A. It must also have a lower resistance than the series combination of bulbs A and B. Bulb C is in parallel with the combination of A and B, so bulb C and the combination of A and B must have the same potential difference across them. Therefore $(P = (\Delta V)^2 / R)$ bulb C must dissipate more power than, and is brighter than, the combination of A and B.

If bulb C is brighter than the combination of A and B, it must be brighter than either A or B alone.

\[ C > A > B \]

4. (5 points) A battery of emf $E$ is connected to the circuit shown. The circuit comprises one capacitor of capacitance $C$ and three identical resistors of resistance $R$. Once the capacitor is fully charged, the switch $S$ is flipped to the open position. What is the time constant $\tau$ associated with the discharge of the capacitor?

The time constant is

\[ \tau = R_{eq}C \]

The three resistors all have the same potential difference across them, and so are in parallel.

\[ R_{eq} = \left( \frac{1}{R} + \frac{1}{R} + \frac{1}{R} \right)^{-1} = R/3 \Rightarrow \tau = RC/3 \]
5. (5 points) A coaxial cable is made of a conducting cylindrical core of radius $R$ carrying a uniform current density magnitude $J_0$ flowing in the $+\hat{i}$ direction. The shell of the cable, of internal radius $2R$ and external radius $3R$ is also conducting and carries a uniform current density magnitude $J_0$ flowing in the $-\hat{i}$ direction. The space between $R$ and $2R$ is empty. Assuming an infinitely long cable, what is the magnitude of the magnetic field at a point $P$ located just outside of the coaxial cable at a distance $3R$ from the cable axis?

Use Ampere’s Law. A concentric circular path through point $P$ will follow a field line so that the magnetic field $\vec{B}$ has constant magnitude, and is parallel to the path of the loop at every point. Letting current into the page be positive, and remembering the relationship between current and current density,

$$I \cdot B \cdot d\vec{s} = \mu_0 I_{\text{thru}} \Rightarrow B \int B \cos \theta \, ds = \mu_0 \left( J_0 A_{\text{shell}} - J_0 A_{\text{core}} \right)$$

$$B \int ds = \mu_0 \left[ J_0 \left( \pi (3R)^2 - (2R)^2 \right) - J_0 \pi R^2 \right]$$

$$B \cdot 2\pi (3R) = \mu_0 J_0 \pi \left[ 9R^2 - 4R^2 - R^2 \right] = \mu_0 J_0 A \pi R^2 \Rightarrow B = \frac{\mu_0 J_0 A \pi R^2}{6\pi R} = \frac{2}{3} \mu_0 J_0 R$$

6. (5 points) Six infinitely long wires each carry identical currents $I_0$ flowing in the directions sketched below. A positive charge is released at the point $P$ and is given a velocity $\vec{v} = v \hat{j}$. What is the direction, if any, of the magnetic force experienced by the particle just after it is released? Note that $\hat{i}$ is directed out of the page.

First, find the direction of the magnetic field produced by the wires at the location of the charged particle. The top wire, carrying current to the left, produces field out of the page. The bottom wire, also carrying current to the left, produces field in to the page. These two wires, then, produce no net field at the point $P$.

The wire just above the point $P$, carrying current out of the page, produces field to the right. The wire just below the point $P$, also carrying current out of the page, produces field to the left. These two wires, then, produce no net field at the point $P$.

The wire to the left of point $P$, carrying current in to the page, and the wire to the right of point $P$, carrying current out of the page, both produce field down the page. The net field at point $P$, then, is down the page.

The force on a charged particle moving in a magnetic field is

$$\vec{F} = I\vec{\ell} \times \vec{B}$$

Using the Right-Hand-Rule, we find that the force is in to the page, which is in the direction $-\hat{i}$.
7. (5 points) A battery of emf $\mathcal{E} = 10\, \text{V}$ is connected to the circuit shown. All resistors are identical with resistance $R = 28\, \Omega$. The point $A$ of the circuit is connected to the ground. What is the value of the electric potential at point $B$, with respect to zero at ground?

From symmetry considerations, each resistor carries the same current $I$, and it is down the page. If one were to move along the circuit down from $A$ and then up to $B$, adding potential changes, first there would be a decrease $IR$, and then an increase $IR$. Points $A$ and $B$ are at the same potential. If point $A$ is defined to be at zero Volts, then point $B$ must also be at 0.

\[ \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.5\textwidth]{circuit.png}}
\end{array}
\end{array} \]

8. (5 points) There is a magnetic field at the origin, pointing in the $+z$ direction. Which particle would produce that field?

Use the Biot-Savart Law

\[ \vec{B} = \frac{\mu_0 q}{4\pi} \frac{\vec{v} \times \hat{r}}{r^2} \]

The vector product requires that the field be perpendicular to both $\vec{v}$ and $\hat{r}$. That excludes particles $i$ and $iii$. Furthermore, no field is produced when $\vec{v}$ and $\hat{r}$ are parallel, excluding particle $v$. The field due to particle $iv$ would be in the $-z$ direction, so the field must be due to $ii$.

\[ \begin{array}{c}
\begin{array}{c}
\text{\includegraphics[width=0.5\textwidth]{magnetic_field.png}}
\end{array}
\end{array} \]

Particle $ii$, which is negative, on the $+y$ axis, and moving in the $+x$ direction.
9. (5 points) A circular loop of diameter $D$ carries a current $I_{loop}$ and is just touching a long straight wire. Both lie in the same plane, but the wire and loop are electrically isolated so that their currents are independent. What direction must the current $I_{wire}$ in the straight wire have in order to create a magnetic field at the center of the loop in the direction opposite to that created by the loop?

Assume the current in the loop is clockwise. The Right-Hand-Rule from the Biot-Savart Law shows that the magnetic field at the center of the loop is in to the page. For the straight wire to create field out of the page at that location, the same Right-Hand-rule shows that the current must be to the left. If the direction of current in the loop is reversed, then the direction of the field is reversed, so the direction of current in the straight wire must be reversed as well.

$I_{wire}$ is to the left if $I_{loop}$ is clockwise, but to the right if $I_{loop}$ is counter-clockwise.

10. (5 points) In the problem above, what current $I_{wire}$ must the long straight wire carry in order to have zero net magnetic field magnitude at the center of the loop?

Finding the contributions to the field at the center of the loop by applying Ampere’s Law to the wire and the Biot-Savart Law to the loop,

\[
\int\vec{B}_{wire} \cdot d\vec{s} = B 2\pi \left( \frac{D}{2} \right) = \mu_0 I_{wire} \quad \Rightarrow \quad B_{wire} = \frac{\mu_0 I_{wire}}{\pi D}
\]

\[
\int d\vec{B}_{loop} = \int \frac{\mu_0 I_{loop}}{4\pi} \frac{d\vec{\ell} \times \hat{r}}{|\vec{r}|^2} \quad \Rightarrow \quad B_{loop} = \frac{\mu_0 I_{loop}}{4\pi (D/2)^2} \int_0^D d\ell = \frac{\mu_0 I_{loop}}{D}
\]

For the field to have zero net magnitude, the field due to the wire and the field due to the loop must add to zero. They must be in opposite directions, as above, and have the same magnitude.

\[
B_{wire} = B_{loop} \quad \Rightarrow \quad \frac{\mu_0 I_{wire}}{\pi D} = \frac{\mu_0 I_{loop}}{D} \quad \Rightarrow \quad I_{wire} = \pi I_{loop}
\]