Physics 2212 GH  
Spring 2015

Solutions

Fundamental Charge \( e = 1.602 \times 10^{-19} \text{ C} \)

Mass of an Electron \( m_e = 9.109 \times 10^{-31} \text{ kg} \)

Coulomb constant \( K = 8.988 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2 \)

Vacuum Permittivity \( \epsilon_0 = 8.854 \times 10^{-12} \text{ C}^2/\text{N} \cdot \text{m}^2 \)

Unless otherwise directed, friction and drag should be neglected.

Any integrals in free-response problems must be evaluated. Questions about magnitudes will state so explicitly.

I. (18 points) Capacitors \( C_1 = 9.0 \mu \text{F} \) and \( C_2 = 12.0 \mu \text{F} \) are each charged with an \( E = 14 \text{ V} \) battery as in (i).

They are then disconnected from the battery without changing the charge on the capacitor plates. The two capacitors are then connected in parallel as in (ii), with the positive plate of \( C_1 \) connected to the negative plate of \( C_2 \) and vice versa. Afterward, what is the charge on each capacitor?

The initial charge on each capacitor can be determined.

\[
\begin{align*}
Q_{1i} &= C_1 \Delta V_1 = (9.0 \mu \text{F})(14 \text{ V}) = 126 \mu \text{C} \\
Q_{2i} &= C_2 \Delta V_2 = (12.0 \mu \text{F})(14 \text{ V}) = 168 \mu \text{C}
\end{align*}
\]

Since the capacitors are reconfigured in (ii) with the original positive plate of one capacitor attached to the negative plate of the other, some of the original charge will "cancel" when the charge is rearranged. The total charge \( Q_{ii} \) on the two capacitors, then, will be the difference between the original charges.

\[
Q_{ii} = Q_{2i} - Q_{1i} = 168 \mu \text{C} - 126 \mu \text{C} = 42 \mu \text{C}
\]

The capacitors in (ii) are in parallel, so the electric potentials across the two capacitors must be the same.

\[
\Delta V_{1ii} = \Delta V_{2ii} \quad \Rightarrow \quad \frac{Q_{1ii}}{C_1} = \frac{Q_{2ii}}{C_2}
\]

Eliminate \( Q_{2ii} \) (for example) and solve for \( Q_{1ii} \).

\[
\begin{align*}
Q_{2ii} &= Q_{ii} - Q_{1ii} \\
\frac{Q_{1ii}}{C_1} &= \frac{Q_{ii} - Q_{1ii}}{C_2} \\
Q_{1ii} C_2 = Q_{ii} - Q_{1ii} \\
\Rightarrow \quad Q_{1ii} + Q_{1ii} \frac{C_2}{C_1} &= Q_{ii} \\
\Rightarrow \quad Q_{1ii} \left( 1 + \frac{C_2}{C_1} \right) &= Q_{ii} \\
Q_{1ii} &= \frac{Q_{ii}}{1 + \frac{C_2}{C_1}} = \frac{42 \mu \text{C}}{1 + (12.0 \mu \text{F})/(9.0 \mu \text{F})} = 18 \mu \text{C}
\end{align*}
\]

Then

\[
Q_{2ii} = Q_{ii} - Q_{1ii} = 42 \mu \text{C} - 18 \mu \text{C} = 24 \mu \text{C}
\]
II. (16 points) The network shown comprises five capacitors of identical capacitance $C = 8.0 \mu F$. An insulating material of dielectric constant $\kappa = 5.0$ is inserted in the rightmost capacitor and fills it entirely. The network is charged by a battery of emf $E = 15 \text{ V}$. Once electrostatic equilibrium is reached the battery is disconnected. How much electric potential energy is stored in the network, with respect to zero when all capacitors are uncharged?

Find the equivalent capacitance of the circuit. Capacitors $C_3$ and $C_5$ are in parallel. Remember that the capacitance of a capacitor increases by a factor of $\kappa$ when filled with a dielectric.

$$C_{35} = C_3 + C_5 = C + \kappa C = (1 + 5) C = 6C$$

Capacitors $C_2$, $C_{35}$, and $C_4$ are in series.

$$C_{2345} = \left( \frac{1}{C_2} + \frac{1}{C_{35}} + \frac{1}{C_4} \right)^{-1} = \left( \frac{1}{C} + \frac{1}{6C} + \frac{1}{C} \right)^{-1} = \frac{6}{13} C$$

Capacitor $C_1$ is in parallel with capacitor $C_{2345}$.

$$C_{12345} = C_1 + C_{2345} = C + \frac{6}{13} C = \frac{19}{13} C$$

So the total energy stored is

$$U_{12345} = \frac{1}{2} C_{12345} (\Delta V_{12345})^2 = \frac{1}{2} \left( \frac{19}{13} C \right) E^2 = \frac{19}{26} (8.0 \mu F)(15 \text{ V})^2 = 1.3 \text{ mJ}$$

Note that the stored energy is not affected by disconnecting the battery after equilibrium has been reached.

1. (5 points) In the above network, what is the ratio of the charges $|Q_2|/|Q_5|$ between the capacitor labeled 2 and the capacitor labeled 5?

Capacitor $C_2$ is in series with the $C_{35}$ combination, so $Q_{35} = Q_2$. Since $Q_3 \neq 0$, $Q_2 > Q_5$, meaning $Q_2/Q_5 > 1$. That excludes three of the five choices offered.

Capacitor $C_3$ and capacitor $C_5$ are in parallel, so they have the same potential difference. Since capacitor $C_5$ is filled with the dielectric, its capacitance is greater than that of $C_3$. The definition of capacitance, $Q = C \Delta V$, shows us $Q_5 > Q_3$. Remembering that $Q_{35} = Q_2$, we now know that $Q_5$ is more than half of $Q_2$, so $Q_2/Q_5 < 2$. This excludes one of the two remaining choices offered.

$$\frac{Q_2}{Q_5} = \frac{6}{5}$$
III. (16 points) You are given a block of aluminum (i) of length $\ell$, height $h$ and square cross-section, and a copper rod (ii) one-third of that length ($\ell/3$) and with diameter $d = h$.

Using an Ohmmeter, you measure an electrical resistance $R_i = 1.6\, \text{k}\Omega$ between the right and the left faces of the aluminum block. Knowing that the electrical conductivity of copper is approximately $\sigma_{\text{Cu}} = 2\sigma_{\text{Al}}$, what resistance $R_{ii}$ would you measure between the right and the left faces of the copper rod?

Remember the relationships between resistance and resistivity, and between resistivity and conductivity.

\begin{align*}
R &= \frac{\rho}{A} \quad \text{and} \quad \sigma = \frac{1}{\rho} \\
R_{\text{Al}} &= \rho_{\text{Al}} \frac{L_{\text{Al}}}{A_{\text{Al}}} = \frac{L_{\text{Al}}}{\sigma_{\text{Al}} A_{\text{Al}}} = \frac{\ell}{\sigma_{\text{Al}} h^2} \\
R_{\text{Cu}} &= \rho_{\text{Cu}} \frac{L_{\text{Cu}}}{A_{\text{Cu}}} = \frac{L_{\text{Cu}}}{\sigma_{\text{Cu}} A_{\text{Cu}}} = \frac{\ell/3}{2\sigma_{\text{Al}} (\pi/4) h^2} = \left( \frac{4}{6\pi} \right) \frac{\ell}{\sigma_{\text{Al}} h^2} = \left( \frac{2}{3\pi} \right) R_{\text{Al}} = \left( \frac{2}{3\pi} \right) 1.6\, \text{k}\Omega = 340\, \Omega
\end{align*}

2. (5 points) You now connect the block and the rod as shown. The right and left faces of this object are connected to a power-supply of emf $E$. What is the ratio of current densities $J_{\text{Al}}/J_{\text{Cu}}$ between the aluminum part and the copper part of the object?

The currents must be the same, so

\begin{align*}
I_{\text{Al}} &= I_{\text{Cu}} \Rightarrow J_{\text{Al}} A_{\text{Al}} &= J_{\text{Cu}} A_{\text{Cu}} \\
J_{\text{Al}} h^2 &= J_{\text{Cu}} \left( \frac{\pi}{4} \right) h^2 \Rightarrow \frac{J_{\text{Al}}}{J_{\text{Cu}}} &= \frac{\pi}{4}
\end{align*}
3. (5 points) An electric potential \( V \) varies with position \( x \), as shown. At which location does the electric field have its greatest magnitude?

Remember the relationship between electric potential and field:

\[ E_x = -\frac{\delta V}{\delta s} \]

So the field with the greatest magnitude \( E_x \) where \( \frac{\delta V}{\delta x} \) has greatest magnitude, that is, where the graph is steepest, which is at location iv.

![Graph of electric potential V vs x with locations marked]

4. (5 points) An electric potential \( V \) varies with position \( x \), as shown. If it can be determined, what is the direction, if any, of the electric field at location ii?

Remember the relationship between electric potential and field:

\[ E_x = -\frac{\delta V}{\delta s} \]

The negative sign indicates that the field is in the opposite direction of the graph’s slope. That slope is positive at location ii, so the field is

In the \(-x\) direction.

![Graph of electric potential V vs x with locations marked]
5. (5 points) A parallel-plate capacitor has plate area $A$ and plate separation $d$. It is charged by connecting it to a battery with emf $\mathcal{E}$. If it is possible to double the electric field magnitude between the plates by changing their geometry, how could that be done?

With the capacitor attached to the battery, the potential difference $\Delta V$ between the plates is fixed. The electric field in a parallel-plate capacitor is uniform, so

$$\Delta V = - \int \vec{E} \cdot d\vec{s} \quad \Rightarrow \quad \Delta V = Ed \quad \Rightarrow \quad E = \frac{\Delta V}{d}$$

therefore, the electric field magnitude can be doubled

By halving the plate separation to $d/2$.

6. (5 points) A parallel-plate capacitor has plate area $A$ and plate separation $d$. It is charged by connecting it to a battery with emf $\mathcal{E}$, then it is disconnected from the battery. If it is possible to double the electric field magnitude between the plates by changing their geometry, how could that be done?

With the capacitor disconnected from the battery, the charge on the plates is fixed. The electric field in a parallel-plate capacitor is

$$E = \frac{\eta}{\epsilon_0} = \frac{Q}{A\epsilon_0}$$

therefore, the electric field magnitude can be doubled

By halving the plate area to $A/2$. 
7. (5 points) The empty parallel-plate capacitor in (i) is charged such that an electric field of magnitude $E_0$ exists between the electrodes separated by a distance $d$. An insulator of dielectric constant $\kappa$ and width $d/2$ is carefully inserted inside the charged capacitor resulting in (ii). What is the magnitude $E$ of the electric field between the electrodes of the capacitor (ii)?

Since the charge on the capacitor plates doesn’t change, the field in the $d/4$ regions outside the dielectric

$$E_{\text{outside}} = \frac{\eta}{\epsilon_0} = \frac{Q}{A\epsilon_0} = E_0$$

doesn’t change. Within the $d/2$ region inside the dielectric, however, the permittivity of the material must be considered.

$$E_{\text{inside}} = \frac{\eta}{\epsilon} = \frac{Q}{A\kappa \epsilon_0} = E_0/\kappa$$

so

$$E = E_0$$ outside the insulator, $$E = E_0/\kappa$$ inside the insulator.

8. (5 points) In the above problem, what is the difference of potential $|\Delta V|$ between the electrodes of the capacitor (ii)?

Remember the relationship between electric potential and field, and that the electric field is uniform in a parallel-plate capacitor.

$$\Delta V = -\int \vec{E} \cdot d\vec{s} \Rightarrow |\Delta V| = E_0 \left( \frac{d}{4} \right) + E_0 \left( \frac{d}{2} \right) + E_0 \left( \frac{d}{4} \right) = E_0 \left( \frac{d}{2} \right) + \frac{E_0}{\kappa} \left( \frac{d}{2} \right)$$

$$= E_0 \left( \frac{d}{2} \right) \left( 1 + \frac{1}{\kappa} \right) = E_0 \left( \frac{d}{2\kappa} \right) (\kappa + 1) = E_0 d \frac{1 + \kappa}{2\kappa}$$
9. (5 points) Two segments of wire have equal diameters but different conductivities \( \sigma_1 \) and \( \sigma_2 \). A current \( I \) passes through the wire. What is the sign of the charge, if any, that lies on the boundary between the wire segments?

Use Gauss’ Law, considering a Gaussian Surface around the boundary. If \( E_1 < E_2 \), then the net flux though the surface is outward (positive), meaning there must be net positive charge at the boundary. But if \( E_1 > E_2 \), then the net flux though the surface is inward (negative), meaning there must be net negative charge at the boundary.

The currents in the two segments must be the same, and since the cross-sectional areas are the same, the current densities must be the same. This allows the electric fields and conductivities to be related

\[
I_1 = I_2 \quad \Rightarrow \quad J_1 A_1 = J_2 A_2 \quad \Rightarrow \quad J_1 = J_2 \quad \Rightarrow \quad \sigma_1 E_1 = \sigma_2 E_2
\]

So

\[
E_1 > E_2 \quad \text{when} \quad \sigma_1 < \sigma_2 \quad \text{and} \quad E_1 < E_2 \quad \text{when} \quad \sigma_1 > \sigma_2
\]

The charge at the boundary, then, is positive if \( \sigma_1 > \sigma_2 \), but negative if \( \sigma_1 < \sigma_2 \).

10. (5 points) An electric field \( \vec{E} \) varies only with position \( x \) as shown. If the electric potential is zero at \( x = 0 \) m, what is the greatest electric potential in the range \( x = 0 \) m to \( x = 12 \) m?

Remember the relationship between electric potential and field

\[
\Delta V = - \int \vec{E} \cdot d\vec{s}
\]

That means the change in electric potential is the opposite of the area under the graph. Since the graph starts from \( x = 0 \) m with positive area beneath it, the electric potential decreases from zero as one moves toward \( x = 6 \) m. From that point to \( x = 12 \) m the potential increases, but one can see from the symmetry of the graph, that it only gets back to zero. That is, the potential is negative everywhere between \( x = 0 \) m and \( x = 12 \) m. Only at \( x = 0 \) m and \( x = 12 \) m does the potential have its maximum value of 0 V.